Pair Dependent Linear Statistics for the Circular β Ensembles

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Objectives

In our paper, Pair Dependent Linear Statistics for CβE [1], we studied the limiting distribution of

$$S_n(f) = \sum_{1\le i\neq j\le n} f(L_n(\theta_i - \theta_j)), \tag{1}$$

where $\theta_1, \dots, \theta_n$ are distributed according to the $C\beta E_n$, L_n is a non-decreasing sequence satisfying $1 \le L_n \le n$ and f is a suitable test function.

Motivation

Our work is largely inspired by that of H. Montgomery [4, 5], which connected the behavior of rescaled non-trivial zeros of the **Riemann zeta function** with the local eigenvalue statistics for the CUE_n. Assuming the Riemann Hypothesis to be true, suppose that $\{1/2 \pm \gamma_n\}$ are the 'non-trivial' zeroes of the Riemann zeta function and consider the scaling $\overline{\gamma_n} = \frac{\gamma_n}{2\pi}\log(\gamma_n)$, so that the spacing between neighboring, rescaled zeros is on the order of a constant. Montgomery essentially studied the statistic

$$S_T(\alpha) = \frac{1}{T}\sum_{0<\overline{\gamma}_j,\overline{\gamma}_k\le T} \frac{\exp(i\alpha(\overline{\gamma}_j - \overline{\gamma}_k))}{1+\left(\frac{\overline{\gamma}_j-\overline{\gamma}_k}{2\log(T)}\right)^2}$$

for real α, and large, real T. Assuming the Riemann Hypothesis, he was able to $S_T(\alpha)$ converges in T to $\min(|\alpha|, 1)$, which is the Fourier transform of

$$\delta(x) - \left(\frac{\sin(\pi x)}{\pi x}\right)^2,$$

the limiting pair correlation function for the local CUE_n eigenvalue statistics.This implies that the local statistics for the zeros of the Riemann zeta function should be the same as those for the CUE_n.

Global Statistics

Theorem 1:
Consider the $C\beta E_n$ and let f be real even function on the unit circle such that $f' \in L^2(\mathbb{T})$ for $\beta = 2$, $\sum_{k\in\mathbb{Z}} |\hat{f}(k)||k| < \infty$ for $\beta < 2, \sum_{k\in\mathbb{Z}} |\hat{f}(k)||k|\log(|k|+1) < \infty$ for $\beta = 4$, and $\sum_{k\in\mathbb{Z}} |\hat{f}(k)||k|^2 < \infty$ for $\beta \in (2,4)\cup(4,\infty)$.
Then we have the following convergence in distribution as $n\to\infty$:

$$S_n(f) - \mathbb{E}S_n(f) \xrightarrow{\mathcal{D}} \frac{4}{\beta}\sum_{k=1}^{\infty} \hat{f}(k)k(\varphi_k - 1),$$

where φ_k are i.i.d. exponential random variables with $\mathbb{E}(\varphi_k) = 1$.

CLT for Mesoscopic Statistics

Theorem 2:
Let $f \in C_c^2(\mathbb{R})$ be an even, smooth, compactly supported function on the real line. Assume that $1 << L_n << n$ for $\beta = 2$ and that L_n grows to infinity slower than any positive power of n for $\beta \neq 2$. Then $(S_n(f(L_n\cdot)) - \mathbb{E}S_n(f(L_n\cdot)))L_n^{-1/2}$ converges in distribution to centered real Gaussian random variable with variance:

$$\frac{4}{\pi\beta^2}\int_{\mathbb{R}} |\hat{f}(t)|^2 t^2 dt,$$

where $\hat{f}$ denotes the Fourier transform of f.

Covariance Structure ($\beta = 2$)

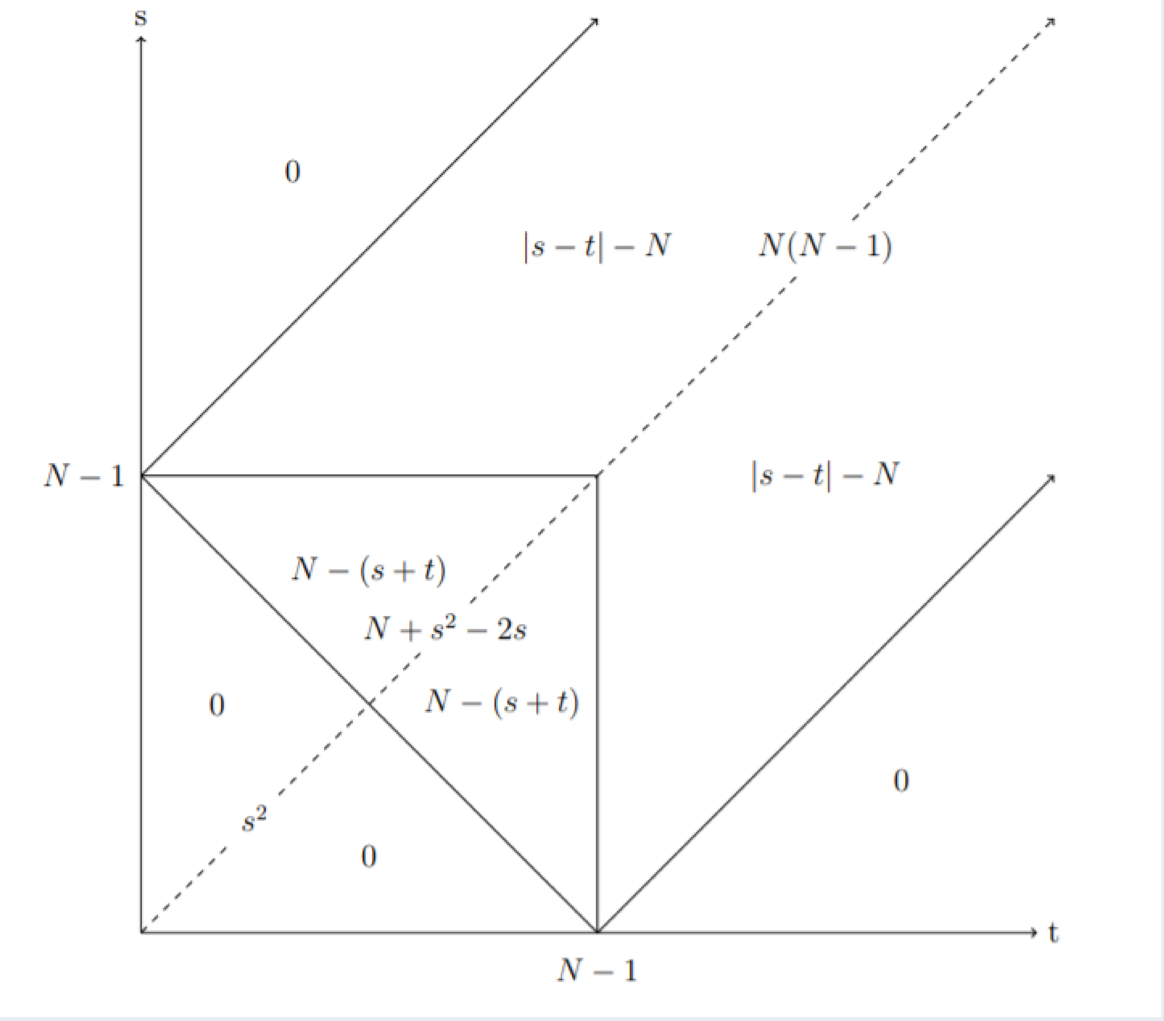

Figure 1: $\mathrm{Cov}\,(|T_{s,n}|^2, |T_{t,n}|^2)$

CLT for Local Statistics ($\beta = 2$)

Theorem 3:
Let $f \in C_c^\infty(\mathbb{R})$ be an even, smooth, compactly supported function on the real line and consider $S_n(f(L_n\cdot))$ with respect to local CUE_n statistics. Then $(S_n(f(n\cdot)) - \mathbb{E}S_n(f(n\cdot)))n^{-1/2}$ converges in distribution to centered real Gaussian random variable with variance:

$$\frac{1}{\pi}\int_{\mathbb{R}} |\hat{f}(t)|^2 \min(|t|,1)^2 dt - \frac{1}{\pi}\int_{|s-t|\le 1, |s|\vee|t|\ge 1} \hat{f}(t)\hat{f}(s)(1-|s-t|)dsdt$$
$$- \frac{1}{\pi}\int_{0\le s,t\le 1, s+t>1} \hat{f}(s)\hat{f}(t)(s+t-1)dsdt.$$

Proof Methods

Global Regime:
$S_n(f) - \mathbb{E}(S_n(f))$ can be rewritten in terms of the powers of traces of $C\beta E_n$ distributed matrices as follows:

$$S_n(f) - \mathbb{E}\left(S_n(f)\right) = \frac{4}{\beta}\sum_{k>1}\hat{f}(k)\left(|T_{k,n}|^2 - \min(k,n)\right) + o_n(1),$$

where $\hat{f}(k)$ denotes the k-th fourier coefficient of $f(L_n\cdot)$ and $T_{k,n}$ denotes the trace of the k-th power of a $C\beta E_n$ distributed matrix. The problem of studying the distribution of $S_n(f)$ is then equivalent to studying the joint distribution of powers of traces of $C\beta E_n$ distributed matrices. When f is a trigonometric polynomial, the result follows from the work of K. Johansson [3].
To consider more general test functions, we used the determinental structure of the 2, 3, and 4-point correlation functions for CUE_n to explicitly compute the variance of $S_n(f)$. The covariance structure for the random variables $\{|T_{k,n}|^2\}_{k\in\mathbb{N}}$ was an immediate corollary (See Figure 1.). The proof is then completed by using a standard argument involving the Lévy metric and the Chebychev inequality. When $\beta \neq 2$, we applied the work of T. Jiang and S. Matsumoto [2] on joint moments for traces of $C\beta E_n$ distributed matrices.
Mesoscopic Regime:
In the Mesoscopic case ($1 << L_n << n$), $\mathrm{Var}(S_n(f))$ is immediately obtained from the Global case by applying the correct renormalization and considering Riemann sums. A standard Lindeberg Feller argument, combined with results about the joint moments of traces, finishes the proof.
Local Regime:
The proof of the CLT for local statistics requires additional work is quite different from the previous two proofs in that it is combinatorial in nature and involves joint cumulants and power counting arguments.

References

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