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Minimal Characteristic Algebras for Rectangular k-Normal Identities*

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Abstract. A characteristic algebra for a hereditary property of identities of a fixed type τ is an algebra $\mathcal{A}$ such that for any variety V of type τ, we have $\mathcal{A} \in V$ if and only if every identity satisfied by V has the property p. This is equivalent to $\mathcal{A}$ being a generator for the variety determined by all identities of type τ which have property p. Płonka has produced minimal (smallest cardinality) characteristic algebras for a number of hereditary properties, including regularity, normality, uniformity, biregularity, right- and leftmost, outermost, and external-compatibility. In this paper, we use a construction of Płonka to study minimal characteristic algebras for the property of rectangular k-normality. In particular, we construct minimal characteristic algebras of type (2) for k-normality and rectangularity for $1 \leq k \leq 3$.

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1 Introduction

Let τ be a fixed type of algebras and identities. A property p of identities of type τ is said to be hereditary if any identity deducible (by the usual rules of deduction) from a set of identities all having the property p must also have the property p. A variety V is said to have property p when all the identities of V have property p. When p is a hereditary property, the set $p(\tau)$ of all identities of type τ having property p is an equational theory, and thus defines a variety V_p which is the smallest variety to have property p.

A characteristic algebra for a hereditary property p is an algebra $\mathcal{A}$ such that for any variety V of type τ, every identity of V has property p if and only if $\mathcal{A}$ is in V. This is equivalent to saying that the set $Id\,\mathcal{A}$ of identities satisfied by $\mathcal{A}$ is

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exactly the set $p(\tau)$ of type τ identities having property p; that is, $\mathcal{A}$ satisfies all identities with property p, but does not satisfy any identity which does not have property p. Therefore, $\mathcal{A}$ being characteristic for p is also equivalent to $\mathcal{A}$ being a generating algebra for the variety V_p.

In [8] and [9], Płonka produced characteristic algebras for a number of hereditary properties of identities, including normality, regularity, biregularity, external-compatibility, uniformity and rectangularity, and for some combinations of these. In [9], the focus was on finding minimal characteristic algebras for these properties, that is, characteristic algebras of the smallest cardinality possible.

Our goal in this paper is to study minimal characteristic algebras for the intersection of two properties, k-normality and rectangularity, for type (2). We will refer to this combined property as rectangular k-normality. We use minimal characteristic algebras known for the individual properties, and a construction of Płonka which combines characteristic algebras for two properties into a characteristic algebra for the intersection property. Our characteristic algebras for k-normality come from the paper [2] of Christie, Wang and Wismath. Massé, Wang and Wismath [6] also used these to find minimal characteristic algebras for the intersection of k-normality and leftmost, using Płonka's construction, and they showed that this construction does not always produce a minimal characteristic algebra. Here we introduce some new techniques from [1] to determine whether the algebras produced by Płonka's construction are minimal or not for type (2) and $k = 1, 2, 3$.

In Section 2, we describe the properties of rectangularity and k-normality for $k \geq 1$, and present minimal characteristic algebras for rectangularity and for 1-, 2- and 3-normality for the special case of type (2). Section 3 applies Płonka's construction for producing characteristic algebras for an intersection of properties to the algebras from Section 2. Sections 4 and 6 describe the two main concepts we use in constructing minimal characteristic algebras, the level of an element in an algebra and the $(k+1)$-level inflation of an algebra, while Section 5 presents a number of necessary conditions for a minimal characteristic algebra. Finally, we exhibit in Section 7 a six-element minimal characteristic algebra for rectangular 2-normality, and in Section 8 a nine-element minimal algebra for $k = 3$.

2 Rectangularity and k-Normality

In this section, we describe the two properties of rectangularity and k-normality, where k is a natural number ≥ 1, and summarize the results on minimal characteristic algebras for these properties.

An identity $s \approx t$ is said to be leftmost if the first variable appearing in s (reading left to right) is the same as the first variable in t. Dually, an identity $s \approx t$ is said to be rightmost when the last (rightmost) variables appearing in s and t are the same. An identity which is both leftmost and rightmost is called outermost, or in type (2) rectangular. It is well known that for type (2), where there is one binary operation symbol denoted simply by juxtaposition, the smallest rectangular variety is the variety RB of rectangular bands, defined by the identities $x(yz) \approx (xy)z$, $x^2 \approx x$ and $x(yz) \approx xz$. It follows from [9, Theorem 2] that the algebra $\mathcal{F}$ with base set $\{p, q, r, s\}$ and binary operation given by the operation table in Figure 2.1

is a minimal characteristic algebra for rectangularity.

F	p	q	r	s
p	p	q	p	q
q	p	q	p	q
r	r	s	r	s
s	r	s	r	s

Figure 2.1. Minimal characteristic algebra $\mathcal{F}$ for rectangularity

An identity $s \approx t$ is said to be normal if either s and t are the same variable, or neither of s or t is a variable. The property of k-normality is a generalization of normality based on the use of the complexity of terms to measure the complexity of identities and hence of varieties. We assume a type $\tau = (n_i)_{i \in I}$ with an n_i-ary operation symbol f_i for each index $i \in I$, and moreover, there are no nullary symbols, so each $n_i \geq 1$. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables. The most commonly used measure of complexity of terms is the depth of a term. For each term t of type τ, we denote by $d(t)$ the depth of t, defined inductively by (i) $d(t) = 0$ if t is a variable x_j for some $j \geq 1$; and (ii) $d(t) = 1 + \max\{d(t_j) : 1 \leq j \leq n_i\}$ if t is a composite term $t = f_i(t_1, \ldots, t_{n_i})$.

When a term t is represented as a tree diagram, its depth corresponds to the length of the longest path from the root to a leaf in the diagram. The depth function is an example of a valuation function on the set of all terms of type τ (see [3]).

Let $k \geq 0$ be a natural number. An identity $s \approx t$ is called k-normal with respect to the depth valuation if either $s = t$ or both $d(s)$ and $d(t)$ are $\geq k$. A non-trivial variety V is called k-normal if all its identities are k-normal, and non-k-normal otherwise. The theory of k-normal varieties with respect to any valuation v was developed by Denecke and Wismath in [3]. They showed that with respect to the usual depth valuation, the k-normality property of identities is hereditary. In addition, for any variety V and $k \geq 0$, the variety $N_k(V)$ determined by all the k-normal identities of V is the least k-normal variety to contain V. Two special cases of this are of interest here. The variety $N_k(TR)$, where TR denotes the trivial variety of type (2), is the smallest k-normal variety of type (2). Applying the k-normalization operator N_k to the variety RB of rectangular bands gives us the variety $N_k(RB)$, the smallest rectangular k-normal variety of type (2), and our goal is to find generators for this variety for $k = 1, 2, 3$.

We note here that the concept of k-normal variety is related to Volkov's definition of a k-nilpotent variety in [10]. However, Volkov used the length of a term as the complexity measurement, rather than the depth, and he defined a variety to be k-nilpotent if it satisfies *all* k-normal identities of its type. In our definition, a variety is k-normal if all of its identities are k-normal.

For $k = 1$ and v the depth valuation, the concepts of k-normal and the k-normalization coincide with the usual concepts of normality and normalization described in [5] and [7]. In [9], Płonka gave a minimal characteristic algebra for normality for an arbitrary type τ. In particular, for type (2), we have a minimal characteristic algebra $\mathcal{A}_1$ for 1-normality, the algebra with a two-element base set $A_1 = \{b, c\}$ with operation defined by $bb = bc = cb = cc = b$.

For $k \geq 1$, it was proven in [2] that a finite minimal characteristic algebra $\mathcal{A}_k$ for k-normality exists for type (2). In this paper, we will use the minimal characteristic algebras of type (2) for $k = 1, 2, 3$. For $k = 2$, we have a size-four minimal characteristic algebra $\mathcal{A}_2$ consisting of the base set $\{b, c, d, e\}$ along with a binary operation defined by the operation table in Figure 2.2.

A_2	b	c	d	e
b	b	b	b	b
c	b	b	b	b
d	b	b	c	b
e	b	b	c	c

A_3	b	c	d	e	f	g	h
b	b	b	b	b	b	b	b
c	b	b	b	b	b	b	b
d	b	b	c	b	c	c	c
e	b	b	c	c	b	b	c
f	b	b	b	c	d	e	d
g	b	b	b	c	d	e	c
h	b	b	c	c	e	e	e

Figure 2.2. Minimal characteristic algebras for 2- and 3-normality

The same figure also displays the operation table for a size-seven minimal characteristic algebra $\mathcal{A}_3$ for type (2) from [2].

3 Płonka's Construction

In [9], Płonka gave a construction which produces a characteristic algebra for the intersection of two properties from individual characteristic algebras. The construction uses two special kinds of elements possible in algebras.

Let $\mathcal{A}$ be an algebra of arbitrary type and let a be an element of A. We say that a is idempotent if $f_i(a, \ldots, a) = a$ for every $i \in I$. The element a is called absorbing if for every $i \in I$ and all $a_1, \ldots, a_n \in A$, we have $f_i(a_1, \ldots, a_n) = a$ whenever $a \in \{a_1, \ldots, a_n\}$. Clearly, any absorbing element is also idempotent, but the converse is not true in general.

Theorem 3.1. [9] *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type. If b is an absorbing element of A and s is an idempotent element of B, then the subdirect product algebra*

$$\mathcal{A} \times_{\langle b,s \rangle} \mathcal{B} = \{(b, q) \mid q \in B\} \cup \{(r, s) \mid r \in A\}$$

satisfies precisely those identities which hold in both $\mathcal{A}$ and $\mathcal{B}$. As a consequence, if $\mathcal{A}$ is characteristic for a property p_1 and $\mathcal{B}$ is characteristic for a property p_2, then $\mathcal{A} \times_{\langle b,s \rangle} \mathcal{B}$ is characteristic for the intersection property $p_1 \cap p_2$.

We can use this theorem to produce characteristic algebras of type (2) for the combined property of being both k-normal and rectangular. We start with the characteristic algebra $\mathcal{F}$ for rectangularity from Figure 2.1 with base set $\{p, q, r, s\}$ and the element s as our distinguished idempotent. For k-normality, we use the finite minimal characteristic algebra $\mathcal{A}_k$ proven to exist in [2]. From the description in [4] of algebras in varieties of the form $N_k(V)$ for some V, we know that there exists an absorbing element $b \in A_k$ for each $k \geq 1$. Applying Płonka's construction thus produces a characteristic algebra $\mathcal{C}_k$ for rectangular k-normality:

$$\mathcal{C}_k = \mathcal{A}_k \times_{\langle b,s \rangle} \mathcal{F} = \{(b, x) \mid x \in F\} \cup \{(y, s) \mid y \in A_k\}$$

with binary operation defined coordinatewise. Since $\{(b,x) \mid x \in F\} \cap \{(y,s) \mid y \in A_k\} = \{(b,s)\}$, the base set of $\mathcal{C}_k$ has cardinality equal to one less than the sum of the cardinalities of A_k and F. To simplify our notation, we denote the element $(b,x) \in C_k$ by x for all $x \in F \setminus \{s\}$, and the element $(y,s) \in C_k$ by y for all $y \in A_k$.

The following examples illustrate the algebras obtained by using Płonka's construction on the algebras from Section 2.

Example 3.2. For $k = 1$, the subdirect product construction using $\mathcal{A}_1$ and $\mathcal{F}$ produces (using our notational simplification) an algebra $\mathcal{C}_1 = \mathcal{A}_1 \times_{\langle b,s \rangle} \mathcal{F}$ with base set $\{p, q, r, b, c\}$ and operation table given in Figure 3.1:

C_1	p	q	r	b	c
p	p	q	p	q	q
q	p	q	p	q	q
r	r	b	r	b	b
b	r	b	r	b	b
c	r	b	r	b	b

Figure 3.1. Characteristic algebra $\mathcal{C}_1$ for rectangular 1-normality

Płonka has shown in [9] that any minimal characteristic algebra for rectangular normality must have size at least five. Hence, the algebra $\mathcal{C}_1$ is a minimal characteristic algebra for rectangular 1-normality.

Example 3.3. Here we apply Płonka's construction to the minimal characteristic algebras $\mathcal{A}_2$ and $\mathcal{A}_3$ from Figure 2.2. Both these algebras contain an absorbing element b, and we use the idempotent $s \in F$, to produce the algebras $\mathcal{C}_2 = \mathcal{A}_2 \times_{\langle b,s \rangle} \mathcal{F}$ and $\mathcal{C}_3 = \mathcal{A}_3 \times_{\langle b,s \rangle} \mathcal{F}$ whose operation tables are shown in Figure 3.2:

C_2	p	q	r	b	c	d	e
p	p	q	p	q	q	q	q
q	p	q	p	q	q	q	q
r	r	b	r	b	b	b	b
b	r	b	r	b	b	b	b
c	r	b	r	b	b	b	b
d	r	b	r	b	b	c	b
e	r	b	r	b	b	c	c

C_3	p	q	r	b	c	d	e	f	g	h
p	p	q	p	q	q	q	q	q	q	q
q	p	q	p	q	q	q	q	q	q	q
r	r	b	r	b	b	b	b	b	b	b
b	r	b	r	b	b	b	b	b	b	b
c	r	b	r	b	b	b	b	b	b	b
d	r	b	r	b	b	c	b	c	c	c
e	r	b	r	b	b	c	c	b	b	c
f	r	b	r	b	b	b	c	d	e	d
g	r	b	r	b	b	b	c	d	e	c
h	r	b	r	b	b	c	c	e	e	e

Figure 3.2. Characteristic algebras for rectangular 2- and 3-normality

Again, it follows from Theorem 3.1 that these two algebras $\mathcal{C}_2$ and $\mathcal{C}_3$ are characteristic for rectangular 2- and 3-normality, respectively. We will show later that neither is minimal. Although we do not use Płonka's construction directly in the proof of minimality, it does give us an upper bound for the minimal size; and in addition, the structure of these two algebras strongly influences our study in Sections 5, 7 and 8 of necessary properties of any (minimal) characteristic algebra.

4 Levels of Elements

In this section, we introduce one of the tools we will need to find minimal characteristic algebras for rectangular k-normality, the concept of the level of an element in an algebra. This is a measurement for complexity of elements in an algebra, analogous to the depth of a term, and was used in [2], [6] and [7].

Definition 4.1. Let $\mathcal{A}$ be an algebra of arbitrary type τ (with no nullary operation symbols), and let k be a fixed natural number. We assign to each $a \in A$ a non-negative integer not exceeding k in the following way. If a is obtainable as the output of the term operation induced on $\mathcal{A}$ by some term t of depth $0 \leq j \leq k-1$, but not for any term of greater depth, then we assign to a the integer j and say that a is at or has *level* j (in $\mathcal{A}$, with respect to k). Whenever a is the output of an induced term operation corresponding to a term of depth k or more, a is assigned the natural number k, and we say that a is at *level* k in $\mathcal{A}$.

We note that any element a of any algebra $\mathcal{A}$ will always have some level ≥ 0 assigned to it in $\mathcal{A}$, because a can always be obtained at least as the output of variable terms of depth 0. Moreover, since there are terms of type τ of arbitrarily high depth, there will be elements in $\mathcal{A}$ of the highest level k. It is also clear that any idempotent element will be at level k, since it is the output of terms of arbitrarily high depth.

We now give a few basic lemmas about the levels of elements in any algebra. The first follows immediately from the definition.

Lemma 4.2. *Let $\mathcal{A}$ be an algebra of arbitrary type τ (with no nullary symbols), and let k be a fixed natural number. If $t = t(x_1, \ldots, x_n)$ is an n-ary term of type τ with depth j, and $a_1, \ldots, a_n \in A$, then the element $t^{\mathcal{A}}(a_1, \ldots, a_n)$ is at level at least j (but not higher than k) in $\mathcal{A}$.*

Lemma 4.3. *Let $\mathcal{A}$ be an algebra of arbitrary type τ (with no nullary symbols), and let k be a fixed natural number. If $a_1, \ldots, a_n$ are elements of A with a_l at level $0 \leq j_l \leq k$ for each $1 \leq l \leq n$, then for any n-ary operation symbol f of type τ, the element $f^{\mathcal{A}}(a_1, \ldots, a_n)$ has level at least $1+\max\{j_1, \ldots, j_n\}$ if $\max\{j_1, \ldots, j_n\} < k$, and has level k if $\max\{j_1, \ldots, j_n\} = k$.*

Proof. Suppose $a_1, \ldots, a_n \in A$ such that a_l has level $0 \leq j_l \leq k$. By definition, for each a_l, there exists an m_l-ary term t_l of type τ with depth j_l and some inputs $u_{l1}, \ldots, u_{lm_l} \in A$ for which $t_l^{\mathcal{A}}(u_{l1}, \ldots, u_{lm_l}) = a_l$. So for any n-ary operation symbol f of type τ, $f^{\mathcal{A}}(a_1, \ldots, a_n) = f^{\mathcal{A}}(t_1^{\mathcal{A}}(u_{11}, \ldots, u_{1m_1}), \ldots, t_n^{\mathcal{A}}(u_{n1}, \ldots, u_{nm_n}))$. Thus, $f^{\mathcal{A}}(a_1, \ldots, a_n)$ is the output of the term $f(t_1, \ldots, t_n)$ for some inputs from A. By definition, $f(t_1, \ldots, t_n)$ has depth equal to $1 + \max\{d(t_l) \mid 1 \leq l \leq n\} = 1 + \max\{\text{level}(a_l) \mid 1 \leq l \leq n\}$, where level$(a_l)$ denotes the level of a_l. The lemma then follows from the definition of level. □

In addition to the depth of a term, we can define the depth of a particular occurrence of a variable in a term, within that term. Using the tree induced by a term t, we define the depth of an occurrence of a variable x_j in t to be the length of the path from the root of the tree to that occurrence of x_j.

Lemma 4.4. *Let $\mathcal{A}$ be an algebra of arbitrary type τ (with no nullary symbols), and let k be a fixed natural number. Let $t = t(x_1, \ldots, x_n)$ be a term of type τ and arity $n \geq 1$. Suppose some variable x_j, where $0 \leq j \leq n$, occurs at depth p in t. Then for all $u_1, \ldots, u_n \in A$, the element $t^{\mathcal{A}}(u_1, \ldots, u_n)$ has level not less than $\min\{p + w, k\}$ whenever u_j is at level w in $\mathcal{A}$.*

Proof. If u_j is at level w, then there exists a term $s(y_1, \ldots, y_m)$ of depth w and inputs $v_1, \ldots, v_m \in A$ such that $s^{\mathcal{A}}(v_1, \ldots, v_m) = u_j$. Hence,

$$t^{\mathcal{A}}(u_1, \ldots, u_j, \ldots, u_n) = t^{\mathcal{A}}(u_1, \ldots, u_{j-1}, s^{\mathcal{A}}(v_1, \ldots, v_m), u_{j+1}, \ldots, u_n).$$

But by the definition of depth, $t(x_1, \ldots, x_{j-1}, s(y_1, \ldots, y_m), x_{j+1}, \ldots, x_n)$ has depth at least $w + p$. The definition of level then implies the desired conclusion. □

It is convenient at this stage to introduce some notation. First, for each $0 \leq j \leq k$, we define $L_j^A = \{a \in A \mid a \text{ has level } j \text{ in } \mathcal{A}\}$. We can now state the following easy corollary of Lemma 4.3.

Corollary 4.5. *Let $\mathcal{A}$ be an algebra of arbitrary type τ (with no nullary symbols), and let k be a fixed natural number. For each $0 \leq p \leq k$, the subset $\bigcup_{j=p}^{k} L_j^A$ of A, along with the operations of $\mathcal{A}$ restricted to it, forms a subalgebra of $\mathcal{A}$.*

Of particular interest will be the subalgebra $\mathcal{L}_k^{\mathcal{A}}$ from Corollary 4.5 with base set L_k^A of level k elements.

5 Necessary Conditions for Characteristic Algebras

In this section, we deduce some properties which must hold in any algebra which is characteristic for rectangular k-normality, in order to deduce some lower bounds for size of such algebras. We assume throughout that our type is (2), with a single binary operation denoted by juxtaposition. We recall that our characteristic algebra must satisfy any rectangular k-normal identity, but must not satisfy any identity which is not rectangular or not k-normal (or neither). We will say that such an identity $s \approx t$ must be broken in our algebra, and refer to elements $a_1, \ldots, a_n$ from our algebra for which $s^A(a_1, \ldots, a_n) \neq t^A(a_1, \ldots, a_n)$ as elements which break the identity.

For any term t, we will call the shape of t the term resulting from t when every variable in t is replaced by the variable x. The binary relationship of "having the same shape as" is an equivalence relation on the set of all terms of a fixed type. For type (2), we define the full-shape unary terms FS^j for $j \geq 0$ inductively as follows: $FS^0(x) = x$, and $FS^{j+1}(x) = FS^j(x)FS^j(x)$. It is clear that $FS^j(x)$ has depth j for $j \geq 1$. We will refer to any term with the same shape as one of these FS^j terms as a full-shape term.

Lemma 5.1. *Let k be a natural number. If $\mathcal{A}$ is a characteristic algebra for rectangular k-normality, then the following conditions must hold:*

(i) *There exist $p, a, b \in L_k^A$ such that $pa \neq pb$.*

(ii) *There exist* $q, c, d \in L_k^A$ *such that* $cq \neq dq$.

Proof. Consider the identities

$$FS^k(x)(FS^{k-1}(x)x) \approx FS^k(x)(FS^{k-1}(x)y),$$
$$(xFS^{k-1}(x))FS^k(x) \approx (yFS^{k-1}(x))FS^k(x).$$

Both are non-rectangular and so must break in $\mathcal{A}$. Therefore, there must be elements $e, f, g, h \in A$ such that $(FS^k)^{\mathcal{A}}(e)((FS^{k-1})^{\mathcal{A}}(e)e) \neq (FS^k)^{\mathcal{A}}(e)((FS^{k-1})^{\mathcal{A}}(e)f)$ and $(g(FS^{k-1})^{\mathcal{A}}(g))(FS^k)^{\mathcal{A}}(g) \neq (h(FS^{k-1})^{\mathcal{A}}(g))(FS^k)^{\mathcal{A}}(g)$. By Lemma 4.2, the elements $(FS^k)^{\mathcal{A}}(e)$, $(FS^{k-1})^{\mathcal{A}}(e)e$, $(FS^{k-1})^{\mathcal{A}}(e)f$, $(FS^k)^{\mathcal{A}}(g)$, $g(FS^{k-1})^{\mathcal{A}}(g)$, and $h(FS^{k-1})^{\mathcal{A}}(g)$ are at level k in $\mathcal{A}$. □

Lemma 5.1 has a simple interpretation in terms of the operation table for the subalgebra $\mathcal{L}_k^{\mathcal{A}}$ of $\mathcal{A}$: it says that the table contains at least one non-constant row and at least one non-constant column.

For the next two lemmas, we recall some background information. In type (2), the variety RB of rectangular bands is the variety determined by the identities $x(yz) \approx (xy)z$, $x^2 \approx x$ and $xyz \approx xy$. It is the join (in the lattice of all type (2) varieties) of the varieties LZ of left-zero algebras and RZ of right-zero algebras. These two varieties are defined by $xy \approx x$ and $xy \approx y$, respectively. It is well known that any algebra in RB is a direct product of an algebra from LZ with one from RZ.

The next lemma is a special case of a result from [1, Lemma 3.1], and gives us strong information about the subalgebra of level k elements in any algebra characteristic for rectangular k-normality.

Lemma 5.2. [1, Lemma 3.1] *Let* k *be a natural number. If* $\mathcal{A}$ *is an algebra in* $N_k(RB)$*, then the subalgebra* $\mathcal{L}_k^{\mathcal{A}}$ *is in* RB.

Lemma 5.3. *Let* k *be a natural number, and let* $\mathcal{A}$ *be a finite characteristic algebra for rectangular* k*-normality. Then the cardinality of* L_k^A *is greater than* 1 *and is not a prime number.*

Proof. By Lemma 5.1, L_k^A is non-empty. Now we proceed by contradiction, supposing that the size of L_k^A is either 1 or a prime number. Since $\mathcal{A} \in N_k(RB)$, Lemma 5.2 implies that $\mathcal{L}_k^{\mathcal{A}}$ is in the variety RB of rectangular bands. Therefore, $\mathcal{L}_k^{\mathcal{A}}$ is isomorphic to a product $\mathcal{A}_L \times \mathcal{A}_R$ for some $\mathcal{A}_L \in LZ$ and $\mathcal{A}_R \in RZ$, making $|L_k^A| = |A_L| \cdot |A_R|$. Then there are two cases to consider. First, if $|A_L| = |L_k^A|$, then $\mathcal{L}_k^{\mathcal{A}} \approx \mathcal{L}_k^{\mathcal{A}}$ and we have $ab = a$ for all $a, b \in L_k^A$; but this contradicts Lemma 5.1. In the second case, if $|A_R| = |L_k^A|$, then $\mathcal{L}_k^{\mathcal{A}} \approx \mathcal{L}_k^{\mathcal{A}}$, whence we have $ab = b$ for any $a, b \in L_k^A$, which is again a contradiction. □

Corollary 5.4. *Let* $k \geq 1$ *be a natural number. Any characteristic algebra for rectangular* k*-normality must contain at least four elements at level* k.

Lemma 5.5. *Let* $k \geq 1$ *be a fixed natural number, and let* $\mathcal{A}$ *be a finite characteristic algebra for rectangular* k*-normality. If* $|L_k^A| = 4$*, then* $\mathcal{L}_k^{\mathcal{A}}$ *is isomorphic to a minimal characteristic algebra for the rectangularity property.*

Proof. As above, we have $\mathcal{L}_k^A \approx \mathcal{A}_L \times \mathcal{A}_R$ for some algebras $\mathcal{A}_L \in LZ$ and $\mathcal{A}_R \in RZ$. Since $|L_k^A| = 4$, there are three cases to consider.

Case 1) $|A_L| = 4$ and $|A_R| = 1$. In this case, we must have $\mathcal{L}_k^{\mathcal{A}} \approx \mathcal{A}_L$. But then $ab = a$ for all $a, b \in L_k^A$, contradicting Lemma 5.1.

Case 2) $|A_L| = 1$ and $|A_R| = 4$. A dual argument to that for case 1) shows that this case is also impossible.

Case 3) $|A_L| = |A_R| = 2$. Since $\mathcal{A}_L \in LZ$ and $\mathcal{A}_R \in RZ, \mathcal{A}_L \times \mathcal{A}_R$ is isomorphic to the algebra with base set $\{p, q, r, s\}$ and one binary operation given by Figure 2.1, which is a type (2) minimal characteristic algebra for rectangularity. □

With these results about the size of the set of level k elements in any finite algebra characteristic for k-normality, we can also deduce some information about the behavior of level k elements.

Lemma 5.6. *Let $k \in N$. Suppose $\mathcal{A}$ is an algebra in the variety $N_k(RB)$. Then for any $a \in A$, a is at level k in $\mathcal{A}$ if and only if a is idempotent.*

Proof. We observed above that any idempotent element of an algebra is at level k. Conversely, if $a \in L_k^A$, then for some arity n, there exists a term t of arity n and depth at least k, and elements $u_1, \ldots, u_n$ of A such that $a = t^{\mathcal{A}}(u_1, \ldots, u_n)$. The identity $t \approx tt$ is then rectangular and k-normal and so holds in $\mathcal{A}$. In particular, we have $a = t^{\mathcal{A}}(u_1, \ldots, u_n) = t^{\mathcal{A}}(u_1, \ldots, u_n)t^{\mathcal{A}}(u_1, \ldots, u_n) = aa$. □

Lemma 5.7. *Let $k \in N$. Let $\mathcal{A}$ be a characteristic algebra for rectangular k-normality. Then A contains $k+1$ distinct elements $a_0, \ldots, a_k$ with a_j at level j for each $0 \leq j \leq k$. Furthermore, $a_{j+1} = a_j a_j$ for each $0 \leq j \leq k-1$.*

Proof. A proof of this result for a characteristic algebra for k-normality is given in [2]. Since the identities used in that proof are also rectangular as well as k-normal, the same proof carries over to our situation. □

6 $(k + 1)$-Level Inflations

Let k be a positive integer. In this section, we use the $(k + 1)$-level inflation construction from [1] to deduce several important properties that any characteristic algebra for rectangular k-normality must possess. First we state the key definition and the results we will need.

Given a base algebra $\mathcal{A}$, an inflation of $\mathcal{A}$ is formed by adding disjoint sets of new elements to the base set A, one set C_a (containing a) for each element a of A. The union of these new sets then forms the base set of a new algebra containing $\mathcal{A}$, in which operations are performed by the rule that any element in the set C_a always acts like a.

For the $(k+1)$-level inflation of an algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$, we start by inflating the set A as above, by adding to each $a \in A$ a set C_a containing a such that for $a \neq b \in A$, the sets C_a and C_b are disjoint. Let $A^* = \bigcup\{C_a \mid a \in A\}$. For each element $c \in A^*$, there is a unique element $\bar{c} \in A$ such that $c \in C_{\bar{c}}$. For each $a \in A$, we will refer to C_a as the class of a. These classes form a partition of A^* which induces an equivalence relation θ on A^*. But in addition to this usual inflation of

$\mathcal{A}$, for each $a \in A$, we partition the set C_a into $k+1$ subclasses or levels C_a^j for $j = 0, 1, \ldots, k$. We impose the restriction that $|C_a^k| \geq 1$, but the other levels may be empty. Thus, $C_a = \bigcup_{j=0}^{k} C_a^j$. We say that the elements of C_a^j are attached to a at level j.

A mapping ψ from the power set of A^* to A^* satisfying $\psi(C_a) \in C_a$ for all $a \in A$ will be called a *θ-choice function*. Our new algebra $\mathcal{A}^*$ will have the inflated set A^* as its universe with operations $f_i^{\mathcal{A}^*}$ for each $i \in I$ defined as follows:

Definition 6.1. Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra with A^* and θ as above. Let ϕ be a θ-choice function such that for any $a \in A$, $\phi(C_a) \in C_a^k$. For each $i \in I$, we define $f_i^{\mathcal{A}^*}$ on A^* by setting for any $a_1, \ldots, a_{n_i} \in A^*$,

$$f_i^{\mathcal{A}^*}(a_1, \ldots, a_{n_i}) = \begin{cases} \phi(C_{f_i^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_{n_i}})}) & \text{if } p \geq k-1, \\ \text{any element of } \bigcup_{j=1+p}^{k} C^j_{f_i^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_{n_i}})} & \text{otherwise,} \end{cases}$$

where $p =$ maximum level of $a_1, \ldots, a_{n_i}$.

The algebra $\mathcal{A}^* = (A^*; (f_i^{A^*})_{i \in I}) = \mathit{Inf}_{k+1}(\mathcal{A}, \theta)$ will be called a *$(k+1)$-level inflation* of $\mathcal{A}$.

Lemma 6.2. [1, Theorem 2.3] *Let V be any variety. Any algebra constructed as a $(k+1)$-level inflation of an algebra $\mathcal{A}$ belonging to V is in $N_k(V)$.*

The following special case of Lemma 6.2 will be invaluable for proving that a particular type (2) algebra is characteristic for rectangular k-normality.

Corollary 6.3. *Let $\mathcal{A}$ be an algebra of type (2). If the subalgebra $\mathcal{L}_k^{\mathcal{A}}$ of $\mathcal{A}$ is in the variety RB, and A is constructed as a $(k+1)$-level inflation of $\mathcal{L}_k^{\mathcal{A}}$, then $\mathcal{A} \in N_k(RB)$.*

Lemma 6.4. [1, Theorem 3.2] *Let V be a variety of type τ and $\mathcal{A}$ be an algebra belonging to $N_k(V)$. Suppose there exists a unary term t of type τ and depth at least k such that the following conditions hold:*

(C1) *For any level k element $a \in A$, $t^{\mathcal{A}}(a) = a$.*

(C2) *For each fundamental operation f_i of type τ, $\mathcal{A}$ satisfies the identity $t(f_i(x_1, \ldots, x_{n_i})) \approx f(t(x_1), \ldots, t(x_{n_i}))$.*

Then $\mathcal{A}$ is a $(k+1)$-level inflation of its subalgebra $\mathcal{L}_k^{\mathcal{A}}$.

Lemma 6.4 and its proof in [1] tell us a great deal about the structure of any characteristic algebra for rectangular k-normality. This is the content of the next lemma.

Lemma 6.5. *Let $\mathcal{A}$ be an algebra belonging to $N_k(RB)$. Then A is constructed as a $(k+1)$-level inflation of its subalgebra $\mathcal{L}_k^{\mathcal{A}}$. Moreover, the equivalence relation θ on A associated with the $(k+1)$-level inflation is a congruence, and induces the partition of A into classes $C_p = \{a \in A \mid (FS^k)^{\mathcal{A}}(a) = p\}$ for each $p \in L_k^A$.*

Proof. By Lemma 6.4, it suffices to show that the type (2), depth k term $FS^k(x)$ satisfies conditions (C1) and (C2) for $\mathcal{A}$. Suppose a is any element at level k in

$\mathcal{A}$. Then by Lemma 5.6, a is idempotent and hence $(FS^k)^{\mathcal{A}}(a) = a$, giving us (C1). Also, $\mathcal{A}$ satisfies the type (2) rectangular and k-normal identity $FS^k(xy) \approx FS^k(x)FS^k(y)$, giving us (C2).

To see that θ is a congruence, suppose $a, b, c, d \in A$ with $a\,\theta b$ and $c\,\theta d$. This means that $(FS^k)^{\mathcal{A}}(a) = (FS^k)^{\mathcal{A}}(b)$ and $(FS^k)^{\mathcal{A}}(c) = (FS^k)^{\mathcal{A}}(d)$, which implies that $(FS^k)^{\mathcal{A}}(a)(FS^k)^{\mathcal{A}}(c) = (FS^k)^{\mathcal{A}}(b)(FS^k)^{\mathcal{A}}(d)$. Since the identity $FS^k(xy) \approx FS^k(x)FS^k(y)$ holds in $\mathcal{A}$, it follows that $(FS^k)^{\mathcal{A}}(ac) = (FS^k)^{\mathcal{A}}(bd)$. But this means exactly that $ac\,\theta bd$. □

We note that for every $p \in L_k^A$, we have $p \in C_p$ since p is idempotent. It is also easy to see that elements $a, b \in A$ will be θ-related if and only if $(FS^k)^{\mathcal{A}}(a) = (FS^k)^{\mathcal{A}}(b)$. We will use this relationship extensively, and use the notation $a \equiv b$ to indicate that $(FS^k)^{\mathcal{A}}(a) = (FS^k)^{\mathcal{A}}(b)$. The next two lemmas provide more information about the behavior of elements in an algebra in $N_k(RB)$ with respect to this relation $\equiv$.

Lemma 6.6. *Let $\mathcal{A}$ be an algebra in the variety $N_k(RB)$, and let a, b and p be elements of A.*

(i) *If a and b are both at level k and $a \equiv b$, then $a = b$.*

(ii) *If p is at level k and $p \equiv a$, then for all $r \in L_k^A \cup L_{k-1}^A$, $pr = ar$ and $rp = ra$.*

(iii) *If p and a have levels k or $k-1$ and $p \equiv a$, then for all $r \in A$, $pr = ar$ and $rp = ra$.*

Proof. (i) Elements at level k are idempotent by Lemma 5.6, so we have $a = (FS^k)^{\mathcal{A}}(a) = (FS^k)^{\mathcal{A}}(b) = b$.

(ii) Let r be at level $k-1$ or k. Since $p \equiv a$ and $\equiv$ is a congruence, we get $pr \equiv ar$ and $rp \equiv ra$. But all four of these elements are in level k, so by (i) in fact $pr = ar$ and $rp = ra$.

(iii) Let r be any element of A, and let $p \equiv a$ both be at level $k-1$ or k. Then for any $r \in A$ at any level, we have $pr \equiv ar$ and $rp \equiv ra$. As in (ii), these elements are all at level k, and by (i) then we have $pr = ar$ and $rp = ra$. □

Lemma 6.7. *Let k be a fixed natural number. If $\mathcal{D}_k$ is a characteristic algebra for rectangular k-normality, then for all elements a, b, c, d in D_k, we have $a(cb) \equiv (ac)b \equiv (ac)(db) \equiv ab$.*

Proof. We will prove $a(cb) \equiv ab$ for all a, b, c in D_k, with the proofs of other claims being very similar. We recall that $a \equiv b$ if and only if $(FS^k)^{D^k}(a) = (FS^k)^{D^k}(b)$. In particular, we have $(FS^k)^{D^k}(a) \equiv a$ and $(FS^k)^{D^k}(b) \equiv b$. Since this relation is a congruence by Lemma 6.5, we therefore have $ab \equiv (FS^k)^{D^k}(a)(FS^k)^{D^k}(b) = (FS^k)^{D^k}(a)[c(FS^k)^{D^k}(b)] \equiv a(cb)$, where the equality holds because $FS^k(x)FS^k(y) \approx FS^k(x)zFS^k(y)$ is a rectangular k-normal identity which holds in $\mathcal{D}_k$. □

7 Size Six for Rectangular 2-Normality

In Section 3, we saw a characteristic algebra for rectangular 2-normality of size seven, produced by Płonka's construction. We can now show that any characteristic algebra must contain at least six elements by Corollary 5.4 and Lemma 5.7. Thus, we have an upper bound of seven and a lower bound of six for the size of a minimal characteristic algebra. In this section, we use the tools we have developed so far to show that there does exist a minimal characteristic algebra for rectangular 2-normality of size six, which is in fact unique up to isomorphism.

Suppose $\mathcal{D}_2$ is a characteristic algebra for rectangular 2-normality. Lemma 5.7 tells us that it must contain three elements, denoted by a_0, a_1, a_2, each of which is the square of the previous one and with a_i being at level i for $0 \leq i \leq 2$. Then by Corollary 5.4, there must be three additional elements, to be denoted by p, q, r, each at level 2 in $\mathcal{D}_2$.

D_2	p	q	r	a_2	a_1	a_0
p	p	q	p	q	q	q
q	p	q	p	q	q	q
r	r	a_2	r	a_2	a_2	a_2
a_2	r	a_2	r	a_2	a_2	a_2
a_1	r	a_2	r	a_2	a_2	a_2
a_0	r	a_2	r	a_2	a_2	a_1

Figure 7.1. Minimal characteristic algebra $\mathcal{D}_2$ for rectangular 2-normality

If $\mathcal{D}_2$ is to have size six, these elements identified so far must be the only elements in D_2. Thus, we have exactly four elements at level 2, and by Lemma 5.5 we know that the subalgebra $\mathcal{L}_2^{\mathcal{D}_2}$ of level 2 elements is isomorphic to the algebra $\mathcal{F}$ from Figure 2.1, which is characteristic for rectangularity. This fact, along with the properties of the elements a_0, a_1, a_2, and Lemmas 6.6 and 6.7, show that $\mathcal{D}_2$ must have the operation table in Figure 7.1.

This shows that (up to isomorphism) only one six-element characteristic algebra is even possible for rectangular 2-normality. It now suffices to prove that the algebra with table given in Figure 7.1 is indeed a characteristic algebra, and hence is a minimal characteristic algebra.

We must show that $\mathcal{D}_2$ satisfies all rectangular 2-normal identities and breaks all identities which are not rectangular or not 2-normal. The claim that $\mathcal{D}_2$ satisfies all rectangular 2-normal identities is equivalent to saying that $\mathcal{D}_2 \in N_2(RB)$, which we can prove using Lemma 6.2 and Corollary 6.3. Thus, our strategy is to show how $\mathcal{D}_2$ may be constructed as a $(2+1)$-level inflation of an algebra in RB.

First, we observe from the operation table for $\mathcal{D}_2$ that $L_2^{D_2} = \{p, q, r, a_2\}$, $L_1^{D_2} = \{a_1\}$, and $L_0^{D_2} = \{a_0\}$. We also note that the subalgebra $\mathcal{L}_2^{\mathcal{D}_2}$ of $\mathcal{D}_2$ is characteristic for rectangularity because it is isomorphic to the algebra $\mathcal{F}$. This means that it is in the variety RB and, also, that it breaks all non-rectangular identities. Incidentally, the latter implies that $\mathcal{D}_2$ also breaks all non-rectangular identities since $\mathcal{L}_k^{\mathcal{A}}$ is a subalgebra of $\mathcal{D}_2$.

Now we will see how to view $\mathcal{D}_2$ as a $(2+1)$-level inflation of $\mathcal{L}_2^{\mathcal{D}_2}$. For each $u \in L_2^{D_2}$ and $0 \leq j \leq k$, put $C_u^j = \{a \in D_2 \mid (FS^2)^{\mathcal{D}_2}(a) = u \text{ and } a \text{ has level } j \text{ in } \mathcal{D}_2\}$ and $C_u = \bigcup_{j=0}^{2} C_u^j$. So for example, we have $C_p^2 = \{p\}$, $C_q^1 = \emptyset$, $C_{a_2}^0 = \{a_0\}$, and $C_{a_2} = \{a_2, a_1, a_0\}$. These C_u for $u \in L_2^{D_2}$ form a partition of D_2, and for any given

$u \in L_2^{D_2}$, these C_u^j form a partition of C_u. It is easy to verify that for all $a, b \in D_2$, if m is the maximum level amongst a and b, then we have

$$ab = (FS^2)^{\mathcal{D}_2}(a)(FS^2)^{\mathcal{D}_2}(b) \in C^2_{(FS^2)^{\mathcal{D}_2}(a)(FS^2)^{\mathcal{D}_2}(b)}$$

if $m \geq k$, while if $m < k$ we get

$$ab = (FS^2)^{\mathcal{D}_2}(a)(FS^2)^{\mathcal{D}_2}(b) \in \bigcup_{j=m+1}^{2} C^j_{(FS^2)^{\mathcal{D}_2}(a)(FS^2)^{\mathcal{D}_2}(b)}.$$

This shows that $\mathcal{D}_2$ is a $(2+1)$-level inflation of $\mathcal{L}_2^{\mathcal{D}_2}$. Thus, by Lemma 6.2, $\mathcal{D}_2$ satisfies all rectangular 2-normal identities.

By our observation above that $\mathcal{L}_2^{D_2}$ and hence $\mathcal{D}_2$ break all non-rectangular identities, it remains to show that $\mathcal{D}_2$ breaks all identities which are rectangular but not 2-normal. These are the rectangular identities $s \approx t$, where at least one of s and t has depth less than 2. Without loss of generality, we assume $d(s) \leq d(t)$ so that the depth of s will never exceed 1. First we note that it is sufficient to consider identities which use only the two variables x, y. This is because any identity in more than two variables will consist of a pair of terms whose shapes are the same as a pair of terms in an identity using one or two variables. So if we can break an identity using at most two variables, then any identity in more than two variables whose terms have the same shapes as the terms in the original identity (respectively) can be broken using some duplicate inputs. There are two cases to consider.

The first case is when the depth of s is strictly less than that of t. We divide this case into two subcases according to whether $d(s) = 0$ or 1. If the depth of s is 0, it is enough to break all identities of the form $x \approx t_1(x)$, where $t_1(x)$ is a non-variable, unary term in x. Using the level 0 input $a_0 \in D_2$ for x, we note that $a_0 \neq t^{\mathcal{D}_2}(a_0)$ for any identity $t(x)$ of the prescribed form. For $d(s) = 1$, we only need to separate the term xy from any term t_2 of depth 2 or more. For any inputs from D_2, Lemma 4.2 tells us the output of t will be at level 2 or higher in $\mathcal{D}_2$. So it is enough to find inputs from D_2 for x and y for which the output of xy is at level 1. Using $x = y = a_0$ will accomplish this for $a_0 a_0 = a_1 \in L_1^{\mathcal{D}_2}$.

The second case involves terms s, t with $d(s) = d(t) < 2$. But no rectangular non-2-normal identities $s \approx t$ exist when $d(s) = d(t) = 0$ or 1. So there is nothing to prove in this case.

The above discussion proves the following theorem:

Theorem 7.1. *The algebra $\mathcal{D}_2$ with base set $\{p, q, r, a_2, a_1, a_0\}$ and binary operation defined by the operation table in Figure 7.1 is (up to isomorphism) the unique minimal characteristic algebra for rectangular 2-normality.*

8 Size Nine for Rectangular 3-Normality

In this section, we show that size nine is minimal for a characteristic algebra for rectangular 3-normality. We begin with two lemmas which further constrain the behavior of elements in a characteristic algebra, culminating in the conclusion that any characteristic algebra must be at least size nine. Then we exhibit a size-nine algebra which we prove is indeed characteristic. This shows that the ten-element algebra $\mathcal{C}_3$ from Figure 3.3, obtained from Płonka's construction, is not minimal.

In the next lemmas, we consider the following three sets of identities:

$$S_1 = \{(xx)(yx),\ (xy)(xx),\ (xy)(yx)\},$$
$$S_2 = \{(xx)(xy),\ (xx)(yy),\ (xy)(xy),\ (xy)(yy)\},$$
$$S_3 = \{x(xx),\ (xx)x,\ (xx)(xx)\}.$$

For each of these sets, and any two terms s and t from the same set, the identity $s \approx t$ is rectangular but not 3-normal, and so must break in a characteristic algebra. We refer to input elements from the algebra which break all possible identities from within one set as "separating" the terms in that set. We now claim that this separating must be done by level 0 elements in a characteristic algebra, which forces there to be at least two elements at level 0.

Lemma 8.1. *If $\mathcal{D}_3$ is a characteristic algebra for rectangular 3-normality, then it must contain at least two level zero elements to break the identities in the sets S_1, S_2 and S_3.*

Proof. Since all the terms in these three sets have depth 2, it follows from Lemma 4.2 that any outputs of any of the terms are at level 2 or 3 in $\mathcal{D}_3$; and if any of the inputs are at level 1 or higher, then by Lemma 4.3 the outputs will be at level 3. Moreover, it follows from Lemma 6.7 that the outputs of all the terms in any one of the sets are equivalent under $\equiv$; terms in S_1 and S_3 output an element equivalent to ww when w is used for x, while terms in S_2 output an element equivalent to wr, when w and r are inputs for x and y, respectively. Since there is only one level 3 element per equivalence class, we will not be able to separate the terms in each set from each other if the output is at level 3. This forces us to have outputs at level 2, which in turn means that we must have inputs at level 0 only. Finally, to separate the terms in S_1, we need two distinct level 0 elements, to use for x and y. □

We can now consider a lower bound on the size of a characteristic algebra for rectangular 3-normality. From Lemma 5.7, there is at least one element at each of levels 0, 1, 2 and 3; from Corollary 5.4 there must be at least four elements at level 3; and from Lemma 8.2 there must be at least two elements at level 0. This gives us a minimum of eight elements with a breakdown of 2, 1, 1, and 4 elements at levels 0, 1, 2, and 3, respectively. We will use the notation 2/1/1/4 to indicate this breakdown of elements into levels. We show next that even this is not enough, and we must have size at least nine.

Lemma 8.2. *If $\mathcal{D}_3$ is a characteristic algebra for rectangular 3-normality with exactly two elements at level 0 and one element at level 2, then it must contain at least two elements at level 1.*

Proof. As we saw in the previous lemma, there must be at least two level 0 elements to separate various sets of terms. One of these elements is the element a_0 from Lemma 5.7 with the property that $a_0a_0 = a_1$, $a_1a_1 = a_2$ and $a_2a_2 = a_3$. The other level 0 element will be denoted by b_0. To see how the two elements a_0 and b_0 are used to separate our three sets of terms, we consider the results obtained by substitution of all possible inputs or pairs of inputs into the terms.

Table 1

Inputs		Outputs		
x	y	$(xx)(yx)$	$(xy)(xx)$	$(xy)(yx)$
a_0	b_0	$(a_0a_0)(b_0a_0)$	$(a_0b_0)(a_0a_0)$	$(a_0b_0)(b_0a_0)$
b_0	a_0	$(b_0b_0)(a_0b_0)$	$(b_0a_0)(b_0b_0)$	$(b_0a_0)(a_0b_0)$

Table 2

Inputs		Outputs			
x	y	$(xx)(xy)$	$(xx)(yy)$	$(xy)(xy)$	$(xy)(yy)$
a_0	b_0	$(a_0a_0)(a_0b_0)$	$(a_0a_0)(b_0b_0)$	$(a_0b_0)(a_0b_0)$	$(a_0b_0)(b_0b_0)$
b_0	a_0	$(b_0b_0)(b_0a_0)$	$(b_0b_0)(a_0a_0)$	$(b_0a_0)(b_0a_0)$	$(b_0a_0)(a_0a_0)$

Table 3

Inputs	Outputs		
x	$(xx)x$	$x(xx)$	$(xx)(xx)$
a_0	$(a_0a_0)a_0$	$a_0(a_0a_0)$	$(a_0a_0)(a_0a_0)$
b_0	$(b_0b_0)b_0$	$b_0(b_0b_0)$	$(b_0b_0)(b_0b_0)$

First we consider Table 3 for the set S_3. We note that from Lemma 6.7, all the elements in the first row are equivalent modulo $\equiv$ to a_3, and are all at level 2 or 3. Since each $\equiv$ class contains exactly one level 3 element, and by assumption the only level 2 element is a_2, this means that all the elements in this row equal either a_2 or a_3. Similarly, all the elements on the second row of the table are at level 2 or 3, and each is equivalent to $b_0b_0 \equiv b_0$.

Now we claim that b_0 and its powers must be equivalent to a_3. If not, b_0 would be equivalent to some other level 3 element, say $p \neq a_3$. Then all the elements in the second row of Table 3 must be at level 2 or 3, and equivalent to p: this forces all three entries to equal p. But then this row contributes nothing to the separation of the terms of S_3 from each other, and we must then be able to entirely separate the three terms on the first row of Table 3. But this is impossible, since all the elements of that row are either a_2 or a_3: we cannot separate three terms with only two possible outcomes. This contradiction forces $b_0 \equiv a_3$.

From this we see that any of the entries in either row, in any of the three tables, is equivalent to a_3, and is at level 2 or 3. As above, this means that all the entries in all three tables are either a_2 or a_3. Moreover, no row in any of the tables can be constant, since as we argued above one row is not sufficient to separate three or four terms when there are only two possible outputs. For Table 2, we can say something stronger: no row can have three entries the same, since we cannot then separate those three on the other row of that table either with only two outputs to use.

From these facts, we can argue that all of b_0b_0, a_0b_0 and b_0a_0 must be at level 1 in our algebra. If b_0b_0 were at a higher level, all the entries in the second row of Table 3 would equal a_3; if a_0b_0 were at level 2 or higher, we would have three a_3 entries for the first row of Table 1; and similarly for b_0a_0, using the second row of Table 2.

This gives us elements a_0a_0, b_0b_0, a_0b_0 and b_0a_0 all in level 1 of our characteristic algebra. But if level 1 contains only one element, which must be a_1, these four elements would all be equal to a_1, and all entries in the three tables would be a_2, with nothing separated. Therefore, there must be at least two elements at level 1. □

At this point, we can increase our lower bound on the size of a minimal characteristic algebra for rectangular 3-normality to nine: we saw that a size-eight algebra has to have level breakdown of 2/1/1/4, which the previous lemma now rules out. In fact, the proof of the lemma can be further strengthened to show that if there are two elements at level 0 and only one at level 2, then the three elements a_0b_0, b_0a_0 and b_0b_0 at level 1 must all be distinct and hence there must be at least three elements at level 1 in this scenario.

This allows us to say more about the possible structure of a size-nine characteristic algebra. Adding a new ninth element to our 2/1/1/4 level breakdown can happen in four ways; but two of these are ruled out immediately by the fact that we cannot have 2/2/1/4 or 2/1/1/5. This leaves two cases only, 2/1/2/4 and 3/1/1/4. The first of these can also be eliminated by further detailed analysis of possibilities as in the proof of Lemma 8.2.

Thus, any characteristic algebra of size nine for rectangular 3-normality must have level breakdown of 3/1/1/4. A further detailed analysis of cases also shows that the three level 0 elements must have certain necessary interactions with each other and with the higher level elements. In this way, we are led to the possible table for a characteristic algebra $\mathcal{D}_3$ of size nine, shown in Figure 8.1:

	p	q	r	a_3	a_2	a_1	a_0	b_0	c_0
p	p	q	p	q	q	q	q	q	q
q	p	q	p	q	q	q	q	q	q
r	r	a_3	r	a_3	a_3	a_3	a_3	a_3	a_3
a_3	r	a_3	r	a_3	a_3	a_3	a_3	a_3	a_3
a_2	r	a_3	r	a_3	a_3	a_3	a_3	a_3	a_3
a_1	r	a_3	r	a_3	a_3	a_2	a_3	a_2	a_3
a_0	r	a_3	r	a_3	a_3	a_2	a_1	a_2	a_1
b_0	r	a_3	r	a_3	a_3	a_3	a_2	a_1	a_2
c_0	r	a_3	r	a_3	a_3	a_3	a_2	a_1	a_2

Figure 8.1. A size-nine algebra for rectangular 3-normality

We now show that $\mathcal{D}_3$ satisfies all rectangular 3-normal identities and breaks all identities which are not rectangular or not 3-normal. First let us note that the subalgebra $\mathcal{L}_3^{D_3}$ consisting of the four level 3 elements is isomorphic to the characteristic algebra $\mathcal{F}$ for rectangularity, which means that elements from it can be used to break any identity which is not rectangular. To see that $\mathcal{D}_3$ satisfies all rectangular 3-normal identities, we show that it is a 4-level inflation of this subalgebra $\mathcal{L}_3^{D_3}$. This can be done much as the analogous proof for $k = 2$ was done in Section 7, and we omit the details. Then by Lemma 6.2, $\mathcal{D}_3$ is in $N_3(RB)$ and so satisfies all rectangular 3-normal identities.

It remains to show that $\mathcal{D}_3$ breaks all identities which are rectangular but not 3-normal. These are the rectangular identities $s \approx t$, where at least one of s and t

has depth less than 3. Without loss of generality, we assume that the depth of s is less than or equal to the depth of t, so the depth of s will never exceed two. As in Section 7, we note that it is also sufficient to consider identities which use only the two variables x, y.

As in the proof for Theorem 7.1, we proceed by cases, depending on whether $d(s) < d(t)$ or $d(s) = d(t)$. In the first case, we look at three subcases determined by the depth of s. If the depth of s is 0, then s is a variable, $= x$ say, and it is enough to break all identities of the form $x \approx t_1(x)$, where $t_1(x)$ is a non-variable unary term in x. We can break such identities using a_0 as the input for x, since as a level 0 element, a_0 is not the output of any non-variable term, while $t_1(a_0)$ will be at level at least 1 by Lemma 4.2.

Next we consider the subcase in which $1 = d(s) < d(t)$. Here we only need to separate the term xy from any term t of depth 2 or more. Using inputs $x = y = a_0$ will accomplish this, since $a_0a_0 = a_1$ is in level 1 but the output of t will be in level at least 2.

The last subcase of the first case occurs when $2 = d(s) < d(t)$. In this case, the output of the term t is always a level 3 element, and it suffices to show that we can find an input for x in any unary term s of depth 2 which gives output a_2, since this output must be at level at least 2 and there is only one level 2 element in our algebra. It is straightforward to check from the table in Figure 8.1 that $a_0(a_0a_0) = (a_0a_0)(a_0a_0) = a_2$, and $(b_0b_0)b_0 = a_2$.

The second case of our proof involves terms s and t, where $d(s) = d(t) < 2$. For depth 0, the only identity possible is $x = x$ by rectangularity, and this is 3-normal by definition. Similarly, the only identities for depth 1 are $xx \approx xx$ and $xy \approx xy$.

This leaves us with the case where $d(s) = d(t) = 2$. Again, we need to consider two subcases. If s and t have different shapes, then it is enough to separate the corresponding unary terms $s(x)$ and $t(x)$. There are three such terms of depth two to examine, the terms $x(xx)$, $(xx)(xx)$ and $(xx)x$ of Table 3. Again, some checking from the operation table of our algebra shows that we can separate each of these three terms from the others: using a_0 for x separates the first two from the third term, while b_0 separates the first from the second.

In the final subcase, we have s and t both of depth 2 and having the same shape. Here we look at all possible rectangular identities, and show that we can break them all. We list all the possibilities below:

i) To separate $x(yx)$ from $x(xx)$, we can use $x = a_0$ and $y = a_3$.

ii) To separate $x(xy)$ from $x(yy)$, we can use $x = a_0$ and $y = b_0$.

iii) To separate $(xy)x$ from $(xx)x$, we can use $x = b_0$ and $y = a_3$.

iv) To separate $(xy)y$ from $(xx)y$, we can use $x = a_0$ and $y = b_0$.

v) To separate the four terms $(xx)(yy)$, $(xy)(xy)$, $(xy)(yy)$ and $(xx)(xy)$ from each other: we use $x = a_0$ and $y = b_0$ to separate the first term from the others; then $x = a_0$ and $y = c_0$ to separate the second and fourth from the third; and finally $x = c_0$ and $y = b_0$ to separate the second and fourth term.

vi) To separate the four terms $(xx)(yx)$, $(xy)(yx)$, $(xy)(xx)$ and $(xx)(xx)$ from each other: we can separate the last term from all the others using $x = a_0$ and $y = a_3$, since the variable y does not occur in any of the others; $x = a_0$ and $y = c_0$

separates the third from the first two; and $x = b_0$ and $y = c_0$ separates the first from the second term.

We have now proved the following theorem:

Theorem 8.3. *The algebra $\mathcal{D}_3$ with base set $\{p, q, r, a_3, a_2, a_1, a_0, b_0, c_0\}$ and binary operation defined by the operation table in Figure 8.1 is a minimal characteristic algebra for rectangular 3-normality.*

For $k = 2$, we saw that the structure of the minimal characteristic algebra was fully constrained, giving us an algebra which is unique up to isomorphism. This is no longer true for $k = 3$. A large number of the entries in the table are constrained, but the six non-diagonal entries in the lower right 3×3 table may be chosen to be any of 180 valid combinations, and the entries for a_1a_0, a_1b_0, a_0a_1 and b_0a_1 can each be either of a_2 or a_3. This gives a total of 2880 possible operation tables (ignoring ordering), which we can show, as we did for Theorem 8.3, are all minimal characteristic algebras. A further analysis may be carried out to calculate the exact number of non-isomorphic such algebras, but we omit the details here for space reasons.

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