Mathematics 101, Section 211  
Midterm Test 1  
February 2, 2011.

Time: 50 minutes.

Books, notes, calculators, cell phones, electronic memory devices, and electronic communication devices are NOT allowed.

Solutions
1. Short-Answer Questions. Put your answers in the boxes provided but show your work also.

Unless otherwise stated, simplify your answers as much as possible. If you need more space, use
the back of the previous page. Parts (a)-(c) are worth one mark each. For parts (d)-(i), each
question is worth 3 marks, but at most one mark will be given for an incorrect answer.

(a) Evaluate \( \int_0^1 \frac{d}{dx} \arctan x \, dx \).

\[ \arctan(1) - \arctan(0) = e^{\arctan(1)} - 1 \]

(b) Evaluate \( \frac{d}{dx} \int_0^1 e^{\arctan x} \, dx \).

Derivative of a
constant is zero

Answer

0

(c) Evaluate \( \frac{d}{dx} \int_0^x e^{\arctan t} \, dt \).

\[ \arctan(x) \]

Answer

\[ e^{\arctan(x)} \]

(d) Find the number \( b \) so that the average value of \( f(x) = 11 + 4x - 3x^2 \) on \( [0, b] \) is equal to 3.

\[ \frac{1}{b-0} \int_0^b (11+4x-3x^2) \, dx = \frac{1}{b} \left( 11b + 2b^2 - b^3 \right) \]

\[ = 11 + 2b - b^3 \]

\[ \text{Fave} = 3 \Rightarrow 11 + 2b - b^3 = 3 \Rightarrow b^3 - 2b - 8 = 0 \]

\( (b-4)(b-2) = 0 \Rightarrow b = 4 \lor b = 2 \)

Answer

\( b = 4 \)

(e) Evaluate \( \int e^{ax} \sqrt{e^{2x} + 1} \, dx \).

\( u = e^{2x} + 1 \]

\[ du = 2e^{2x} \, dx \]

\[ e^{2x} = u-1 \]

\[ e^{4x} = (u-1)^2 = u^2 - 2u + 1 \]

\( \Rightarrow \int \frac{e^{6x}}{\sqrt{e^{2x} + 1}} \, dx = \frac{1}{2} \int \frac{u^2 - 2u + 1}{\sqrt{u}} \, du = \frac{1}{2} \int (u^{3/2} - 2u^{1/2} + u^{-1/2}) \, du \]

\[ = \frac{1}{2} \left( \frac{2}{5} u^{5/2} - 2 \cdot \frac{2}{3} u^{3/2} + 2u^{1/2} \right) + C \]

Answer

\[ \frac{1}{5} (e^{2x} + 1)^{5/2} - \frac{2}{3} (e^{2x} + 1)^{3/2} + 2(e^{2x} + 1)^{1/2} + C \]
(f) The given integral represents the volume of a solid. Describe the solid.

\[ \pi \int_0^1 (y^2 - y^4) \, dy \]

\[ A(y) = \pi (y^3 - \frac{y^6}{3}) \]

Answer:

It is the solid obtained by rotating the region bounded by the curves \( x = y \), \( x = y^4 \), \( y = 0 \), \( y = 1 \) about the \( y \)-axis.

(g) Assume \( f \) is an even function. Write

\[ \int_{-3}^{-1} f(x) \, dx + \int_{1}^{3} f(x) \, dx - \int_{5}^{6} f(x) \, dx + \int_{-3}^{2} f(x) \, dx \]

as a single integral in the form \( \int_{a}^{b} f(x) \, dx \).

Note

\[ \int_{1}^{3} f(x) \, dx = \int_{3}^{1} f(x) \, dx, \quad -\int_{5}^{6} f(x) \, dx = \int_{6}^{5} f(x) \, dx, \quad \text{and} \]

\[ \int_{-3}^{2} f(x) \, dx = \int_{2}^{-3} f(x) \, dx \quad \text{[because \( f \) is even].} \]

\[ \therefore \int_{-3}^{3} f(x) \, dx + \int_{1}^{3} f(x) \, dx - \int_{5}^{6} f(x) \, dx + \int_{-3}^{2} f(x) \, dx = \int_{-1}^{6} f(x) \, dx \]

Answer:

\[ \int_{-1}^{6} f(x) \, dx \]

(h) Find the derivative of the function \( g(x) = \int_{e^{2x}}^{e^{5x}} \frac{\sin t}{\sqrt{1 + t^2}} \, dt \).

\[ g'(x) = \frac{d}{dx} \left( \int_{e^{2x}}^{e^{5x}} \frac{\sin t}{\sqrt{1 + t^2}} \, dt \right) = \frac{d}{dx} \left( \int_{e^{2x}}^{e^{5x}} \frac{\sin t}{\sqrt{1 + t^2}} \, dt - \int_{0}^{x^2+2x} \frac{\sin t}{\sqrt{1 + t^2}} \, dt \right) \]

\[ = \frac{\sin(e^{5x})}{\sqrt{1 + (e^{5x})^2}} \cdot \frac{d}{dx} (e^{5x}) - \frac{\sin(x^2+2x)}{\sqrt{1 + (x^2+2x)^2}} \cdot \frac{d}{dx} (x^2+2x) \]

Answer:

\[ \frac{\sin(e^{5x}) \cdot 5e^{5x}}{\sqrt{1 + e^{10x}}} - \frac{\sin(x^2+2x)}{\sqrt{1 + x^4 + 4x^3 + 4x^2}} \cdot (2x + 2) \]}
(i) A bacteria population starts with 100 million \((10^8)\) bacteria and increases at a rate of \(r(t) = (\ln 2) \cdot 10^8 \cdot 2^t\) bacteria per hour. How many bacteria are there are after 3 hours?

Let \(N(t)\) be the number of bacteria at time \(t\) (with \(t\) measured in hours).

We are given that \(N(0) = 10^8\).

By the Net Change Theorem,

\[
N(3) - N(0) = \int_0^3 r(t) \, dt = (\ln 2) \cdot 10^8 \cdot \left[ \frac{2^t}{\ln 2} \right]_0^3 = 10^8 \cdot (2^3 - 2^0) = 7 \cdot 10^8
\]

\[
N(3) = N(0) + 7 \cdot 10^8 = 10^8 + 7 \cdot 10^8 = 8 \cdot 10^8
\]

**Answer**

**Full-Solution Problems.** In questions 2–5, justify your answers and show all your work.

If a box is provided, write your final answer there. If you need more space, use the back of the previous page. Unless otherwise indicated, simplification of answers is not required.

2. A spherical tank of radius 5 m is half full of oil that has density 1500 kg/m³. Set up a definite integral giving the work required to pump the oil up out of the top of the tank. Do NOT evaluate the integral. Note: \(1500 \times 9.8 = 14700\).

![Diagram of a spherical tank](image)

Note: If you set up your coordinates in a different way, your formulas will look slightly different.

\[ r = 5 - (5 - x^*) \]

We divide the oil, which extends from \(x = 0\) to \(x = 5\), into \(n\) layers of thickness \(\Delta x = \frac{5 - 0}{n}\), and calculate the work in each layer:

\[ W_i = \text{force} \times \text{distance} \]

The volume \(V_i\) of the \(i\)th layer is approximated by the volume of a circular disk of thickness \(\Delta x\) and radius \(r = \sqrt{5^2 - (5 - x^*)^2}\).

Volume of \(i\)th layer = \(V_i = \pi r^2 \Delta x = \pi \left(5^2 - (5 - x^*)^2\right) \Delta x\)

Mass of \(i\)th layer = \(m_i = \text{density} \cdot V_i = 1500 \cdot V_i\)

The force that needs to be exerted on the \(i\)th layer is equal and opposite to the force of gravity on the layer:

\[ F_i = m_i g = 1500 g V_i \]

The \(i\)th layer needs to be raised a distance \(10 - x^*\).

The work to pump out \(i\)th layer = \(W_i = F_i \cdot (10 - x^*) = 1500 g V_i \cdot (10 - x^*)\)

\[ \approx 1500 g \pi \left(5^2 - (5 - x^*)^2\right) \cdot (10 - x^*) \Delta x \]

Work to pump all the oil out = \(W = \lim_{n \to \infty} \sum_{i=1}^{n} 1500 g \pi \left(5^2 - (5 - x^*)^2\right) \cdot (10 - x^*) \Delta x = \int_0^5 1500 g \pi \left(5^2 - (5 - x^*)^2\right) \cdot (10 - x^*) \, dx \)

\[ \cdot \]

\[ W = \lim_{n \to \infty} \sum_{i=1}^{n} \]
3. Evaluate
\[
\lim_{n \to \infty} \frac{1}{n^2} \left( \sqrt{n^2 - 1^2} + \sqrt{n^2 - 2^2} + \cdots + \sqrt{n^2 - n^2} \right)
\]
by first expressing it as a definite integral and then interpreting the integral in terms of the area of a region. Sketch the region.

The limit is
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \sqrt{n^2 - i^2} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} \sqrt{n^2 - i^2} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} \sqrt{1 - \left( \frac{i}{n} \right)^2}.
\]

With \( \Delta x = \frac{b-a}{n} = \frac{1}{n} \), \( a = 0 \), \( b = 1 \), and \( x_i = a + i \Delta x = \frac{i}{n} \), we see that
\[
\lim_{n \to \infty} \frac{1}{n} \cdot \sum_{i=1}^{n} \Delta x \cdot \sqrt{1 - x_i^2} = \lim_{n \to \infty} \frac{1}{n} \cdot \sum_{i=1}^{n} \sqrt{1 - x_i^2} = \int_{0}^{1} \sqrt{1 - x^2} \, dx.
\]

The integral represents the area of the quarter of the circle with radius 1 and center \((0,0)\) which lies in the first quadrant.

\[
\therefore \lim_{n \to \infty} \frac{1}{n} \cdot \sum_{i=1}^{n} \sqrt{n^2 - i^2} = \frac{\pi}{4}.
\]

4. Consider a solid whose base is a circular disk of radius 1 and whose cross sections perpendicular to the base are squares. Set up a definite integral giving the volume of the solid. Do NOT evaluate the integral.

Find the area \(A(x)\) of the cross section.

\[
A(x) = \omega^2 = 4(1-x^2)
\]

\[
V = \int_{-1}^{1} A(x) \, dx = 4 \int_{-1}^{1} (1-x^2) \, dx = 8 \int_{0}^{1} (1-x^2) \, dx
\]
5. Make a labelled sketch of the region enclosed by the given curves. Decide whether to integrate with respect to \( x \) or \( y \). Then find the area of the region.

\[ y = \frac{\pi}{4} x, \quad y = \arctan x \]

\[ y = \frac{\pi}{4} x \text{ or } x = \frac{\pi}{4} y \]

\[ y = \arctan x \text{ or } x = \tan y \]

**Integrating in \( x \):**

\[ A = \int_{-\frac{\pi}{4}}^{0} \left( \frac{\pi}{4} x - \arctan x \right) \, dx + \int_{0}^{\frac{\pi}{4}} \left( \arctan x - \frac{\pi}{4} x \right) \, dx \]

You can integrate \( \arctan x \) using integration by parts, but that wasn't expected of you.

**Integrating in \( y \):**

\[ A = \int_{-\frac{\pi}{4}}^{0} (\tan y - \frac{\pi}{4} y) \, dy + \int_{0}^{\frac{\pi}{4}} \left( \frac{\pi}{4} y - \tan y \right) \, dy \]

\[ \tan y \, dy = \int \frac{\sin y}{\cos y} \, dy = \int -\frac{1}{u} \, du = -\ln |u| + C = -\ln |\cos y| + C \]

[with the substitution \( u = \cos y \), \( du = -\sin y \, dy \)]

\[ A = \left[ -\ln |\cos y| - \frac{\pi}{4} y^2 \right]_{-\frac{\pi}{4}}^{0} + \left( \frac{\pi}{4} y^2 + \ln |\cos y| \right)_{0}^{\frac{\pi}{4}} \]

\[ = (-\ln |\cos (0)|) - \frac{\pi}{4} (0)^2 - (-\ln |\cos (-\frac{\pi}{4})| - \frac{\pi}{4} (-\frac{\pi}{4})^2) \]

\[ + \left( \frac{\pi}{4} (\frac{\pi}{4})^2 + \ln |\cos (\frac{\pi}{4})| \right) - \left( \frac{\pi}{4} (0)^2 + \ln |\cos (0)| \right) \]

\[ = (-0 - 0) - (-\ln (\frac{\pi}{4^2}) - \frac{\pi}{8}) + \left( \frac{\pi}{8} + \ln (\frac{1}{\sqrt{2}}) \right) - (0 + 0) \]

\[ = \frac{\pi}{4} + 2\ln (\frac{1}{\sqrt{2}}) = \frac{\pi}{4} - \ln 2 \]

We can use symmetry to speed up the calculation.

\[ A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left| \frac{\pi}{4} y - \tan y \right| \, dy = 2\int_{0}^{\frac{\pi}{4}} \left| \frac{\pi}{4} y - \tan y \right| \, dy \quad [ \text{because } f(y) = \left| \frac{\pi}{4} y - \tan y \right| \text{ is an even function}] \]

\[ = 2\int_{0}^{\frac{\pi}{4}} (\frac{\pi}{4} y - \tan y) \, dy \]

\[ = 2 \left( \frac{\pi}{8} + \ln (\frac{1}{\sqrt{2}}) \right) \quad \text{by (half) the calculation above.} \]