Mathematics 101, Section 211
Midterm Exam 2 - Sample 1
March 16, 2011.

Time: 50 minutes.

Books, notes, calculators, cell phones, electronic memory devices, and electronic communication devices are NOT allowed.

Justify your answers and show all your work. If you need more space on a question, use the back of the page which preceeds that question. Unless otherwise indicated, simplification of answers is not required.
1. Evaluate $\int_0^1 x \arctan x \, dx$

Integrate by parts.

\[ u = \arctan x \quad dv = x \, dx \]
\[ du = \frac{1}{1 + x^2} \, dx \quad v = \frac{1}{2} x^2 \]
\[ \int u \, dv = uv - \int v \, du \]

\[ \int_0^1 x \arctan x \, dx = \frac{1}{2} x^2 \arctan x \bigg|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1 + x^2} \, dx \]
\[ = \frac{1}{2} x^2 \arctan x \bigg|_0^1 - \frac{1}{2} \int_0^1 \left( 1 - \frac{1}{1 + x^2} \right) \, dx \]
\[ = \frac{1}{2} x^2 \arctan x \bigg|_0^1 - \frac{1}{2} (x - \arctan x) \bigg|_0^1 \]
\[ = \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \left( 1 - \frac{\pi}{4} \right) = \frac{\pi}{4} - \frac{1}{2} \]

2. Evaluate $\int x^2 \sqrt{9 - x^2} \, dx$.

We use trigonometric substitution. From the triangle we see that

\[ \frac{x}{3} = \sin \theta, \quad \frac{\sqrt{9 - x^2}}{3} = \cos \theta, \quad \frac{dx}{3} = \cos \theta \, d\theta. \]

Hence

\[ \int x^2 \sqrt{9 - x^2} = \int (3 \sin \theta)^2 \cdot 3 \cos \theta \cdot 3 \cos \theta \, d\theta \]
\[ = 81 \int \sin^2 \theta \cos^2 \theta \, d\theta \]
\[ = 81 \int \frac{1}{2} (1 - \cos 2\theta) \frac{1}{2} (1 + \cos 2\theta) \, d\theta \]
\[ = \frac{81}{4} \int (1 - \cos^2 2\theta) \, d\theta \]
\[ = \frac{81}{4} \int \left( 1 - \frac{1}{2} (1 + \cos 4\theta) \right) \, d\theta \]
\[ = \frac{81}{4} \int \left( \frac{1}{2} - \frac{1}{2} \cos 4\theta \right) \, d\theta \]
\[ = \frac{81}{8} \left( \theta - \frac{1}{4} \sin 4\theta \right) + C \]
\[ = \frac{81}{8} \left( \theta - \frac{1}{2} \sin 2\theta \cos 2\theta \right) + C \]
\[ = \frac{81}{8} \left( \theta - \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) \right) + C \]
\[ = \frac{81}{8} \left( \theta - \sin \theta \cos \theta (1 - 2\sin^2 \theta) \right) + C \]
\[ = \frac{81}{8} \left( \arcsin \left( \frac{x}{3} \right) - \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} \left( 1 - \frac{2x^2}{9} \right) \right) + C \]
3. Find the limit of the sequence or show that it is divergent.

\[ a_n = \frac{n^n}{n!} \]

We write

\[ a_n = \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} = \frac{n}{1} \left( \frac{n \cdot n \cdots n}{2 \cdot 3 \cdots n} \right) \]

For \( n \geq 2 \), the expression in parentheses (\( \frac{n}{1} \cdots n \)) is greater than or equal to 1 because the numerator is greater than or equal to the denominator. Thus

\[ a_n \geq \frac{n}{1} \cdot 1 = n \quad \text{for} \ n \geq 2 \]

So as \( n \to \infty \), \( a_n \) is forced off to \( \infty \). Hence the sequence diverges.

4. Evaluate \( \int \tan^2 x \sec x \, dx \).

An even power of tangent appears with an odd power of secant, so we should first express the integrand completely in terms of \( \sec x \). Then we can use the formula

\[ \int \sec x \, dx = \ln|\sec x + \tan x| + C \] (\( \star \))

along with integration by parts for higher odd powers of \( \sec x \). (Look at your notes or page 464 of the textbook for the derivation of the formula (\( \star \)).)

First we write

\[ \int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx \] (\( \star \star \))

Now we integrate \( \int \sec^3 x \, dx \) by parts.

\[ u = \sec x \quad dv = \sec^2 x \, dx \quad du = \sec x \tan x \, dx \quad v = \tan x \]

\[ \int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx \]

\[ = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \]

\[ = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \]

Solving for \( \int \sec^3 x \, dx \) gives

\[ \int \sec^3 x \, dx = \frac{1}{2} \left( \sec x \tan x + \int \sec x \, dx \right) \]

Then substituting into (\( \star \star \)) and using the formula (\( \star \)) yields

\[ \int \tan^2 x \sec x \, dx = \frac{1}{2} \left( \sec x \tan x - \int \sec x \, dx \right) = \frac{1}{2} (\sec x \tan x - \ln|\sec x + \tan x|) + C \]
5. Evaluate \( \int \frac{2x^2 - 10x + 13}{x^2 - 6x + 9} \, dx \).

We will use partial fractions, but since the degree of the numerator is \( \geq \) the degree of denominator, we must first perform long division.

\[
\begin{align*}
\frac{2}{x^2 - 6x + 9} &= \frac{2x^2 - 10x + 13}{x^2 - 6x + 9} \\
&= \frac{2x^2 - 10x + 13}{-2x^2 + 12x - 18} \\
&= \frac{2x - 5}{x^2 - 6x + 9} \\
&= \frac{2}{x^2 - 6x + 9} + \frac{2x - 5}{x^2 - 6x + 9} \\
&= \frac{2}{x^2 - 6x + 9} + \frac{2x - 5}{(x-3)^2}
\end{align*}
\]

So we see that

\[
\int \frac{2x^2 - 10x + 13}{x^2 - 6x + 9} \, dx = \int \left( \frac{2}{x^2 - 6x + 9} + \frac{2x - 5}{(x-3)^2} \right) \, dx = 2x + \int \frac{2x - 5}{(x-3)^2} \, dx
\]

Now we find the needed partial fraction decomposition:

\[
\frac{2x - 5}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}
\]

\[
2x - 5 = A(x-3) + B
\]

Equating coefficients gives

\[
2 = A \\
-5 = -3A + B \quad \Rightarrow \quad B = -5 + 3(2) = 1
\]

Thus

\[
\int \frac{2x^2 - 10x + 13}{x^2 - 6x + 9} \, dx = 2x + \int \left( \frac{2}{x-3} + \frac{1}{(x-3)^2} \right) \, dx = 2x + 2 \ln |x-3| - \frac{1}{x-3} + C
\]

5. Find the sum of the series or show that it is divergent.

\[
\sum_{n=1}^{\infty} \ln \left( \frac{n^2}{n^2 + 2n + 1} \right)
\]

We write

\[
\sum_{n=1}^{\infty} \ln \left( \frac{n^2}{n^2 + 2n + 1} \right) = \sum_{n=1}^{\infty} \ln \left( \frac{n^2}{(n+1)^2} \right) = \sum_{n=1}^{\infty} \left[ \ln(n^2) - \ln((n+1)^2) \right]
\]

and notice that the series is telescoping. Indeed, the \( n \)th partial sum of the series is

\[
s_n = \sum_{i=1}^{n} \left[ \ln(i^2) - \ln((i+1)^2) \right]
\]

\[
= [\ln(1^2) - \ln(2^2)] + [\ln(2^2) - \ln(3^2)] + [\ln(3^2) - \ln(4^2)] + \cdots + [\ln(n^2) - \ln((n+1)^2)]
\]

\[
= \ln 1 - \ln((n+1)^2)
\]

\[
= -2 \ln(n+1)
\]

Since \( \lim_{n \to \infty} s_n = \lim_{n \to \infty} -2 \ln(n+1) = -\infty \), we conclude that the series diverges.

Note that the \( n \)th term of the series \( a_n = \ln \left( \frac{n^2}{n^2 + 2n + 1} \right) \) approaches 0 as \( n \to \infty \). So the Divergence Test is not applicable.
7. Evaluate the integral or show that it is divergent.
\[
\int_1^\infty \frac{x + 1}{x^2 + 2x} \, dx
\]
Since
\[
\frac{x + 1}{x^2 + 2x} = \frac{x}{x^2} = \frac{1}{x}
\]
for large \( x \), we expect the given integral to diverge just like the integral \( \int_1^\infty \frac{1}{x} \, dx \) (the p-integral with \( p = 1 \)). To prove this, we observe that
\[
\frac{x + 1}{x^2 + 2x} \geq \frac{x}{x^2 + 2x} \geq \frac{x}{2x^2 + 2x} = \frac{1}{4x}
\]
Then since \( \frac{1}{4} \int_1^\infty \frac{1}{x} \, dx \) diverges, the Comparison Test tells us that \( \int_1^\infty \frac{x + 1}{x^2 + 2x} \, dx \) diverges as well, just as we suspected.

Alternatively, we compute
\[
\lim_{t \to \infty} \int_1^t \frac{x + 1}{x^2 + 2x} \, dx = \lim_{t \to \infty} \frac{1}{2} \int_3^{t^2 + 2t} \frac{u}{u} \, du = \left[ \frac{u}{u} = \frac{x^2 + 2x}{2x + 2} \right]_{u = 2x + 2}^{u = (x + 1)dx} = \frac{1}{2} \lim_{t \to \infty} (\ln(t^2 + 2t) - \ln 3) = \infty
\]
and hence conclude that the integral diverges.

8. Use Simpson’s Rule with \( n = 6 \) to approximate \( \int_0^3 \frac{1}{1 + y^2} \, dy \).

Simpson’s rule says
\[
\int_a^b f(x) \, dx \approx S_n = \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n) \right].
\]
We have
\[
f(x) = \frac{1}{1 + x^2}, \quad \Delta x = \frac{b - a}{n} = \frac{3 - 0}{6} = \frac{1}{2}, \quad x_i = a + i\Delta x = \frac{i}{2}
\]
\[
x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1, \quad x_3 = \frac{3}{2}, \quad x_4 = 2, \quad x_5 = \frac{5}{2}, \quad x_6 = 3.
\]
\[
f(x_0) = \frac{1}{1 + 0^2} = 1, \quad f(x_1) = \frac{1}{1 + \left(\frac{1}{2}\right)^2} = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5}, \quad f(x_2) = \frac{1}{1 + 1^2} = \frac{1}{2},
\]
\[
f(x_3) = \frac{1}{1 + \left(\frac{3}{2}\right)^2} = \frac{1}{1 + \frac{9}{4}} = \frac{4}{13}, \quad f(x_4) = \frac{1}{1 + 2^2} = \frac{1}{5},
\]
\[
f(x_5) = \frac{1}{1 + \left(\frac{5}{2}\right)^2} = \frac{1}{1 + \frac{25}{4}} = \frac{4}{29}, \quad f(x_6) = \frac{1}{1 + 3^2} = \frac{1}{10}.
\]
Therefore
\[
\arctan(3) = \int_0^3 \frac{1}{1 + y^2} \, dy = \int_0^3 f(x) \, dx \approx \frac{1}{3} \left[ 1 + 4 \cdot \frac{4}{5} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{4}{13} + 2 \cdot \frac{1}{5} + 4 \cdot \frac{4}{29} + \frac{1}{10} \right]
\]
9. Find the centroid of the region bounded by the curves \( y = e^x \), \( y = e^{-x} \), \( x = 0 \), and \( x = 1 \).

The area of the region is

\[
A = \int_0^1 (e^x - e^{-x}) \, dx = (e^x + e^{-x}) \bigg|_0^1 = e + e^{-1} - 2
\]

The \( x \)-coordinate of the centroid is given by

\[
\bar{x} = \frac{1}{A} \int_0^1 x (e^x - e^{-x}) \, dx = \frac{1}{A} \left( \int_0^1 e^x \, dx - \int_0^1 x e^{-x} \, dx \right)
\]

We compute both integrals on the right via integration by parts. For the first integral take

\[
\begin{align*}
& u = x & dv = e^x \, dx \\
& du = dx & v = e^x
\end{align*}
\]

Then

\[
\int_0^1 e^x \, dx = e^x \bigg|_0^1 = 1
\]

For the second integral take

\[
\begin{align*}
& u = x & dv = e^{-x} \, dx \\
& du = dx & v = -e^{-x}
\end{align*}
\]

Then

\[
\int_0^1 x e^{-x} \, dx = -xe^{-x} \bigg|_0^1 + \int_0^1 e^{-x} \, dx = -e^{-x} \bigg|_0^1 = 1 - 2e^{-1}
\]

So we get

\[
\bar{x} = \frac{1}{A} \left( 1 - (1 - 2e^{-1}) \right) = \frac{2e^{-1}}{e + e^{-1} - 2}
\]

The \( y \)-coordinate of the centroid is

\[
\bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2} (e^{2x} - e^{-2x}) \, dx = \frac{1}{2A} \left( \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right) \bigg|_0^1
\]

\[
= \frac{1}{4A} (e^2 + e^{-2} - 2) = \frac{e^2 + e^{-2} - 2}{4(e + e^{-1} - 2)}
\]

So the centroid is

\[
\left( \frac{2e^{-1}}{e + e^{-1} - 2}, \frac{e^2 + e^{-2} - 2}{4(e + e^{-1} - 2)} \right)
\]
10. A tank contains 2000 L of water with 30 kg of dissolved sodium fluoride (NaF). Pure water enters the tank at a rate of 20 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much sodium fluoride is in the tank after 2 hours?

Let \( y(t) \) be the amount of NaF in the tank at time \( t \). Then

\[
\frac{dy}{dt} = \text{(rate in)} - \text{(rate out)}
\]

with

\[
\begin{align*}
\text{(rate in)} &= \left(20 \text{ L/min}\right) \left(0 \text{ kg L}^{-1}\right) = 0 \text{ kg L}^{-1} \\
\text{(rate out)} &= \left(20 \text{ L/min}\right) \left(\text{concentration of salt in the tank in kg/L at time } t\right) \\
&= \left(20 \text{ L/min}\right) \left(\frac{y(t) \text{ kg}}{2000 \text{ L}}\right) = \frac{y(t) \text{ kg}}{100 \text{ L}}
\end{align*}
\]

So we have

\[
\frac{dy}{dt} = -\frac{y(t)}{100}
\]

We are given that \( y(0) = 30 \), so we can exclude the solution \( y(t) = 0 \). Writing the equation in terms of differentials and integrating gives

\[
\int \frac{dy}{y} = -\int \frac{dt}{100}
\]

\[
\ln |y| = -\frac{t}{100} + C
\]

Since \( y(0) = 30 \), we have \( \ln 30 = -\frac{0}{100} + C \), and so \( C = \ln 30 \). Thus

\[
\ln |y| = -\frac{t}{100} + \ln 30
\]

\[
|y| = e^{-t/100} \cdot 30
\]

\[
|y| = 30e^{-t/100}
\]

\[
y = \pm 30e^{-t/100}
\]

Since \( y(0) = 30 > 0 \), we have the positive sign choice:

\[
y(t) = 30e^{-t/100}
\]

So the amount of sodium fluoride in the tank after 2 hours is

\[
y(60) = 30e^{-120/100} = 30e^{-6/5} \text{ kg}
\]