6. Let $u = 1/x$. Then $du = -1/x^2 \, dx$ and $1/x^2 \, dx = -du$, so
\[
\int \frac{\sec^2(1/x)}{x^2} \, dx = \int \sec^2 u \, (-du) = -\tan u + C = -\tan (1/x) + C.
\]

20. Let $u = ax + b$. Then $du = a \, dx$ and $dx = (1/a) \, du$, so
\[
\int \frac{dx}{ax + b} = \int \frac{(1/a) \, du}{u} = \frac{1}{a} \int \frac{1}{u} \, du = \frac{1}{a} \ln |u| + C = \frac{1}{a} \ln |ax + b| + C.
\]

28. Let $u = \tan^{-1} x$. Then $du = \frac{dx}{1 + x^2}$, so
\[
\int \frac{\tan^{-1} x}{1 + x^2} \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{(\tan^{-1} x)^2}{2} + C.
\]

30. Let $u = \ln x$. Then $du = (1/x) \, dx$, so
\[
\int \frac{\sin(\ln x)}{x} \, dx = \int \sin u \, du = -\cos u + C = -\cos(\ln x) + C.
\]

32. Let $u = e^x + 1$. Then $du = e^x \, dx$, so
\[
\int \frac{e^x}{e^x + 1} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln(e^x + 1) + C.
\]

37. Let $u = \sin x$. Then $du = \cos x \, dx$, so
\[
\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\sin x| + C.
\]

42. Let $u = x^2$. Then $du = 2x \, dx$, so
\[
\int \frac{x}{1 + x^2} \, dx = \int \frac{1}{2} \frac{du}{1 + u^2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1} (x^2) + C.
\]

58. Let $u = -x^2$, so $du = -2x \, dx$. When $x = 0, u = 0$; when $x = 1, u = -1$. Thus,
\[
\int_0^1 x e^{-x^2} \, dx = \int_0^{-1} e^u \left(-\frac{1}{2} \, du\right) = -\frac{1}{2} \left[e^u\right]_0^{-1} = -\frac{1}{2} (e^{-1} - e^0) = \frac{1}{2} (1 - 1/e).
\]

60. $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6} \, dx = 0$ by Theorem 7(b), since $f(x) = \frac{x^2 \sin x}{1 + x^6}$ is an odd function.

64. Assume $\alpha > 0$. Let $u = a^2 - x^2$, so $du = -2x \, dx$. When $x = 0, u = a^2$; when $x = a, u = 0$. Thus,
\[
\int_0^a x \sqrt{a^2 - x^2} \, dx = \int_0^{a^2} u^{1/2} \left(-\frac{1}{2} \, du\right) = -\frac{1}{4} \int_0^{a^2} u \, du = \frac{1}{2} \int_0^{\alpha^2} \left[2u^{3/2} - 2u^{1/2}\right]_0^{a^2} = \frac{1}{2} a^3.
\]

66. Let $u = 1 + 2x$, so $x = \frac{1}{2} (u - 1)$ and $du = 2 \, dx$. When $x = 0, u = 1$; when $x = 4, u = 9$. Thus,
\[
\int_0^4 \frac{x \, dx}{\sqrt{1 + 2x}} = \int_1^9 \frac{1}{\sqrt{u}} \, \frac{du}{2} = \frac{1}{2} \int_1^9 \left[u^{1/2} - u^{-1/2}\right] \, du = \frac{1}{2} \left[\frac{3}{2} u^{3/2} - 2u^{1/2}\right]_1^9 = \frac{1}{4} \cdot \frac{3}{2} \left[u^{3/2} - 3u^{1/2}\right]_1^9 = \frac{1}{4} \left[(27 - 9) - (1 - 3)\right] = \frac{20}{6} = \frac{10}{3}
\]

70. Let $u = 2\pi t/T - \alpha$, so $du = \frac{2\pi}{T} \, dt$. When $t = 0, u = -\alpha$; when $t = T/2, u = \pi - \alpha$. Thus,
\[
\int_{-\alpha}^{\pi - \alpha} \sin \left(\frac{2\pi}{T} \, t - \alpha\right) \, dt = \int_{-\alpha}^{\pi - \alpha} \sin \left(\frac{T}{2\pi} \cdot \frac{2\pi}{T} t\right) \, dt = \frac{T}{2\pi} \left[-\cos \frac{\pi - \alpha}{\pi} \cdot -\cos \frac{\alpha}{\pi}\right] = \frac{T}{2\pi} \left[-\cos \frac{\pi}{2} \cos \frac{\pi - \alpha}{2}\right] = -\frac{T}{2\pi} \cos \frac{\alpha}{2}
\]

84. Let $u = x + c$. Then $du = dx$, so
\[
\int_a^b f(x + c) \, dx = \int_{a+c}^{b+c} f(u) \, du = \int_{a+c}^{b+c} f(x) \, dx
\]

From the diagram, we see that the equality follows from the fact that we are translating the graph of $f$, and the limits of integration, by a distance $c$. 
Section 6.1

2. \[ A = \int_0^2 \left( \sqrt{x + 2} - \frac{1}{x + 1} \right) dx = \left[ \frac{2}{3}(x + 2)^{3/2} - \ln(x + 1) \right]_0^2 \]
   \[ = \left[ \frac{2}{3}(4)^{3/2} - \ln 3 \right] - \left[ \frac{2}{3}(2)^{3/2} - \ln 1 \right] = \frac{16}{3} - \ln 3 - \frac{4}{3} \sqrt{2} \]

4. \[ A = \int_0^3 [(2y - y^2) - (y^2 - 4y)] dy = \int_0^3 (-2y^2 + 6y) dy = \left[ -\frac{2}{3}y^3 + 3y^2 \right]_0^3 = (-18 + 27) - 0 = 9 \]

6. \[ A = \int_0^{\pi/2} (e^x - \sin x) dx = \left[ e^x + \cos x \right]_0^{\pi/2} \]
   \[ = (e^{\pi/2} + 0) - (1 + 1) = e^{\pi/2} - 2 \]

10. \[ 1 + \sqrt{x} = \frac{3 + x}{3} = 1 + \frac{x}{3} \Rightarrow \sqrt{x} = \frac{x}{3} \Rightarrow x = \frac{x^2}{9} \Rightarrow 9x - x^2 = 0 \Rightarrow x(9 - x) = 0 \Rightarrow x = 0 \text{ or } 9, \text{ so} \]
    \[ A = \int_0^9 \left[ (1 + \sqrt{x}) - \left( \frac{3 + x}{3} \right) \right] dx = \int_0^9 \left[ (1 + \sqrt{x}) - \left( \frac{3 + x}{3} \right) \right] dx \]
    \[ = \int_0^9 (\sqrt{x} - \frac{1}{3}x) dx = \left[ \frac{2}{3}x^{3/2} - \frac{1}{6}x^2 \right]_0^9 = 18 - \frac{27}{2} = \frac{9}{2} \]

18. \[ A = \int_{-3}^3 |(8 - x^2) - x^2| dx = 2 \int_0^3 |8 - 2x^2| dx \]
    \[ = 2 \int_0^2 (8 - 2x^2) dx + 2 \int_2^3 (2x^2 - 8) dx \]
    \[ = 2[8x - \frac{2}{3}x^3]_0^2 + 2[\frac{2}{3}x^3 - 8x]_2^3 \]
    \[ = 2[16 - \frac{16}{3} - 0] + 2[(18 - 24) - (\frac{16}{3} - 16)] \]
    \[ = 32 - \frac{32}{3} + 20 - \frac{32}{3} = 52 - \frac{64}{3} = \frac{92}{3} \]
20. \(4x + x^2 = 12 \iff (x + 6)(x - 2) = 0 \iff x = -6 \text{ or } x = 2\), so \(y = -6 \text{ or } y = 2\) and

\[A = \int_{-6}^{2} \left[ -\frac{1}{2}y^2 + 3 - y \right] dy = \left[ -\frac{1}{12}y^3 - \frac{1}{2}y^2 + 3y \right]_{-6}^{2} = \left(-\frac{7}{3} - 2 + 6\right) - \left(-18 - 18\right) = 22 - \frac{7}{3} = \frac{64}{3}\.

22. \(A = 2 \int_{0}^{1} \left[ \sin\left(\frac{\pi x}{2}\right) - x \right] dx\)

\[= 2 \left\{ -\frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2} \right\}_{0}^{1} = 2 \left[ \left(0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} - 0\right) \right] = \frac{10}{\pi} - 1\]

28. The curves \(y = 3x^2\) and \(y = -4x + 4\) intersect

when \(3x^2 = -4x + 4\) \([\text{for } x \geq 0] \iff 3x^2 + 4x - 4 = 0 \iff (3x - 2)(x + 2) = 0 \Rightarrow x = \frac{2}{3}\). The curves \(y = 8x^2\) and \(y = -4x + 4\) intersect when \(8x^2 = -4x + 4\) \([\text{for } x \geq 0] \iff 8x^2 + 4x - 4 = 0 \iff 2x^2 + x - 1 = 0 \Rightarrow (2x - 1)(x + 1) = 0 \Rightarrow x = \frac{1}{2}

\[A = \int_{0}^{1/2} \left(8x^2 - 3x^2\right) dx + \int_{1/2}^{2/3} \left[-4x^2 + 3x^2 - 3x^2\right] dx = \int_{0}^{1/2} 5x^2 dx + \int_{1/2}^{2/3} (-x^3 + 4x^2 + 4x) dx = \left[\frac{5}{3}x^3\right]_{0}^{1/2} + \left[\frac{2}{3}x^3 + \frac{8}{3}x^2 + 2x\right]_{1/2}^{2/3} = \frac{5}{24} - \frac{8}{3} - \frac{5}{6} + \frac{8}{3} + \frac{8}{3} + \frac{1}{2} - 2 = \frac{45}{216} - \frac{64}{216} - \frac{57}{216} + \frac{27}{216} + \frac{128}{216} - \frac{432}{216} = \frac{58}{216} = \frac{17}{34} = 0.315\]

32. The curves intersect when \(\sqrt{x + 2} = x \Rightarrow x + 2 = x^2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = -1 \text{ or } 2. [-1 \text{ is extraneous}]

\[A = \int_{0}^{4} \left| \sqrt{x + 2} - x \right| dx = \int_{0}^{2} (\sqrt{x + 2} - x) dx + \int_{2}^{4} (x - \sqrt{x + 2}) dx = \left[\frac{2}{3}(x + 2)^{3/2} - \frac{1}{2}x^2\right]_{0}^{2} + \left[\frac{2}{3}x^3 - \frac{2}{3}(x + 2)^{3/2}\right]_{2}^{4} = \left(\frac{16}{3} - 2\right) - \left[\frac{2}{3}(2\sqrt{2}) - 0\right] + \left[2 - \frac{2}{3}(6\sqrt{6})\right] - \left(2 - \frac{16}{3}\right) = 4 + \frac{16}{3} - \frac{4}{3}\sqrt{2} - 4\sqrt{6} = \frac{44}{3} - 4\sqrt{6} - \frac{4}{3}\sqrt{2} \right]_0^{4}.\]
45. We know that the area under curve $A$ between $t = 0$ and $t = x$ is $\int_0^x v_A(t) \, dt = s_A(x)$, where $v_A(t)$ is the velocity of car $A$ and $s_A$ is its displacement. Similarly, the area under curve $B$ between $t = 0$ and $t = x$ is $\int_0^x v_B(t) \, dt = s_B(x)$.

(a) After one minute, the area under curve $A$ is greater than the area under curve $B$. So car $A$ is ahead after one minute.

(b) The area of the shaded region has numerical value $s_A(1) - s_B(1)$, which is the distance by which $A$ is ahead of $B$ after 1 minute.

(c) After two minutes, car $B$ is traveling faster than car $A$ and has gained some ground, but the area under curve $A$ from $t = 0$ to $t = 2$ is still greater than the corresponding area for curve $B$, so car $A$ is still ahead.

(d) From the graph, it appears that the area between curves $A$ and $B$ for $0 \leq t \leq 1$ (when car $A$ is going faster), which corresponds to the distance by which car $A$ is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time $x$ where the area between the curves for $1 \leq t \leq x$ (when car $B$ is going faster) is the same as the area for $0 \leq t \leq 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

50. (a) We want to choose $a$ so that

$$\int_1^a \frac{1}{x^2} \, dx = \int_1^a \frac{1}{x} \, dx \Rightarrow \left[-\frac{1}{x}\right]_1^a = \left[-\frac{1}{x}\right]_1^4 \Rightarrow -\frac{1}{a} + 1 = -\frac{1}{4} + 1 \Rightarrow \frac{5}{4} = \frac{2}{a} \Rightarrow a = \frac{8}{5}.$$

(b) The area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$ is $\frac{3}{4}$ [take $a = 4$ in the first integral in part (a)]. Now the line $y = b$ must intersect the curve $x = 1/\sqrt{y}$ and not the line $x = 4$, since the area under the line $y = 1/4^2$ from $x = 1$ to $x = 4$ is only $\frac{3}{16}$, which is less than half of $\frac{3}{4}$. We want to choose $b$ so that the upper area in the diagram is half of the total area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$. This implies that

$$\int_b^1 (1/\sqrt{y} - 1) \, dy = \frac{1}{2} \cdot \frac{3}{4} \Rightarrow [2 \sqrt{y} - y]_b^1 = \frac{3}{8} \Rightarrow 1 - 2 \sqrt{b} + b = \frac{3}{8} \Rightarrow$$

$$b - 2 \sqrt{b} + \frac{5}{8} = 0. \text{ Letting } c = \sqrt{b}, \text{ we get } c^2 - 2c + \frac{5}{8} = 0 \Rightarrow 8c^2 - 16c + 5 = 0. \text{ Thus, } c = \frac{16 \pm \sqrt{256 - 160}}{16} = 1 \pm \frac{\sqrt{6}}{4}. \text{ But } c = \sqrt{b} < 1 \Rightarrow c = 1 - \frac{\sqrt{6}}{4} \Rightarrow b = c^2 = 1 + \frac{3}{8} - \frac{\sqrt{6}}{2} = \frac{1}{8} (11 - 4 \sqrt{6}) \approx 0.1503.$$

Section 6.2

4. A cross-section is a disk with radius $\sqrt{25 - x^2}$, so its area is $A(x) = \pi (\sqrt{25 - x^2})^2$.

$$V = \int_2^4 A(x) \, dx = \int_2^4 \pi (\sqrt{25 - x^2})^2 \, dx = \pi \int_2^4 (25 - x^2) \, dx = \pi [25x - \frac{1}{3}x^3]_2^4 = \pi [(100 - \frac{64}{3}) - (50 - \frac{8}{3})] = \frac{94}{3} \pi.$$
8. A cross-section is a washer with inner radius $\frac{1}{4}x^2$ and outer radius $5 - x^2$, so its area is

$$A(x) = \pi (5 - x^2)^2 - \pi \left( \frac{1}{4}x^2 \right)^2$$

$$= \pi (25 - 10x^2 + x^4 - \frac{1}{16}x^4).$$

$$V = \int_{-2}^{2} A(x) \, dx = \int_{-2}^{2} \pi (25 - 10x^2 + \frac{15}{16}x^4) \, dx$$

$$= 2\pi \int_{0}^{2} (25 - 10x^2 + \frac{15}{16}x^4) \, dx$$

$$= 2\pi [25x - \frac{10}{3}x^3 + \frac{3}{16}x^5]_0^2 = 2\pi (50 - \frac{80}{3} + 6) = \frac{179}{3}\pi$$

10. A cross-section is a washer with inner radius $x = 2\sqrt{y}$ and outer radius 2, so its area is

$$A(y) = \pi \left[ (2)^2 - \left(2\sqrt{y}\right)^2 \right]$$

$$= \pi (4 - 4y) = 4\pi (1 - y).$$

$$V = \int_{0}^{1} A(y) \, dy = \int_{0}^{1} 4\pi (1 - y) \, dy$$

$$= 4\pi \left[ y - \frac{1}{2}y^2 \right]_0^1 = 4\pi \left[ (1 - \frac{1}{2}) - 0 \right] = 2\pi$$

12. A cross-section is a washer with inner radius $2 - 1$ and outer radius $2 - e^{-x}$, so its area is

$$A(x) = \pi \left[ (2 - e^{-x})^2 - (2 - 1)^2 \right] = \pi \left[ (4 - 4e^{-x} + e^{-2x}) - 1 \right]$$

$$= \pi (3 - 4e^{-x} + e^{-2x}).$$

$$V = \int_{0}^{3} A(x) \, dx = \int_{0}^{3} \pi (3 - 4e^{-x} + e^{-2x}) \, dx$$

$$= \pi \left[ 3x + 4e^{-x} - \frac{1}{2}e^{-2x} \right]_0^3$$

$$= \pi \left[ 6 + 4e^{-2} - \frac{1}{2}e^{-4} - (0 + 4 - \frac{1}{2}) \right]$$

$$= \left( \frac{3}{2} + 4e^{-2} - \frac{1}{2}e^{-4} \right)\pi$$

14. $$V = \int_{1}^{3} \pi \left\{ \left( \frac{1}{x} - (-1) \right)^2 - [0 - (-1)]^2 \right\} \, dx$$

$$= \pi \int_{1}^{3} \left( \frac{1}{x^2} + \frac{2}{x} + 2 \right) \, dx$$

$$= \pi \left. \left( \frac{1}{x} + \ln x \right) \right|_{1}^{3}$$

$$= \pi \left( \frac{1}{3} + \ln 3 \right) - \left( -1 + 0 \right)$$

$$= \pi \left( \frac{2}{3} + \ln 3 \right) = 2\pi (\ln 3 + \frac{1}{3})$$

16. $y = \sqrt{x}$ implies $x = y^2$, so the outer radius is $2 - x^2$.

$$V = \int_{0}^{1} \pi \left[ (2 - x^2)^2 - (2 - y^2)^2 \right] \, dy$$

$$= \pi \int_{0}^{1} \left[ (4 - 4y^2 + y^4) - (4 - 4y + y^2) \right] \, dy$$

$$= \pi \int_{0}^{1} (y^4 - 5y^2 + 4y) \, dy$$

$$= \pi \left[ \frac{1}{5}y^5 - \frac{5}{3}y^3 + 2y^2 \right]_0^1$$

$$= \pi \left( \frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8}{15}\pi$$
34. \[ V = \pi \int_{0}^{\pi} \left\{ \sin x - (-2)^2 - 0 - (-2)^2 \right\} \, dx \]
\[ = \pi \int_{0}^{\pi} \left\{ (\sin x + 2)^2 - 2^2 \right\} \, dx \]

52. An equation of the line is \[ x = \frac{\Delta x}{\Delta y} \cdot y \] + (x-intercept) \[ = \frac{a/2 - b/2}{h - 0} \cdot y + \frac{a - b}{2h} \cdot \frac{y + b}{2} \]
\[ V = \int_{0}^{h} A(y) \, dy = \int_{0}^{h} (2x)^2 \, dy \]
\[ = \int_{0}^{h} \left[ 2 \left( \frac{a - b}{2h} \cdot y + \frac{b}{2} \right) \right]^2 \, dy = \int_{0}^{h} \left[ \frac{a - b}{h} \cdot y + b \right]^2 \, dy \]
\[ = \int_{0}^{h} \left[ \frac{(a - b)^2}{h^2} \cdot y^2 + \frac{2b(a - b)}{h} \cdot y + b^2 \right] \, dy \]
\[ = \frac{(a - b)^2}{3h^2} \cdot y^3 + \frac{b(a - b)}{h} \cdot y^2 + b^2 \cdot y \bigg|_{0}^{h} \]
\[ = \frac{1}{3} (a - b)^2 h + b(a - b) h + b^2 h = \frac{1}{3} (a^2 - 2ab + b^2 + 3ab) h \]
\[ = \frac{1}{3} (a^2 + ab + b^2) h \]

[Note that this can be written as \( \frac{1}{3} (A_1 + A_2 + \sqrt{A_1 A_2}) h \), as in Exercise 50.]

If \( a = b \), we get a rectangular solid with volume \( b^2 h \). If \( a = 0 \), we get a square pyramid with volume \( \frac{1}{3} b^2 h \).

59. The cross-section of the base corresponding to the coordinate \( x \) has length \( y = 1 - x \). The corresponding square with side \( s \) has area \( A(x) = s^2 = (1 - x)^2 = 1 - 2x + x^2 \). Therefore,
\[ V = \int_{0}^{1} A(x) \, dx = \int_{0}^{1} (1 - 2x + x^2) \, dx \]
\[ = \left[ x - x^2 + \frac{1}{3} x^3 \right]_{0}^{1} = (1 - 1 + \frac{1}{3}) - 0 = \frac{1}{3} \]

Or:
\[ \int_{0}^{1} (1 - x)^2 \, dx = \int_{1}^{0} u^2 (-du) \quad [u = 1 - x] = \left[ \frac{1}{3} u^3 \right]_{0}^{1} = \frac{1}{3} \]

60. The cross-section of the base corresponding to the coordinate \( y \) has length \( 2x = 2\sqrt{1 - y} \). \[ y = 1 - x^2 \iff x = \pm \sqrt{1 - y} \] The corresponding square with side \( s \) has area \( A(x) = s^2 = (2\sqrt{1 - y})^2 = 4(1 - y) \). Therefore,
\[ V = \int_{0}^{1} A(y) \, dy = \int_{1}^{0} 4(1 - y) \, dy = 4 \left[ y - \frac{1}{2} y^2 \right]_{0}^{1} = 4 \left[ (1 - \frac{1}{2}) - 0 \right] = 2. \]

66. Each cross-section of the solid \( S \) in a plane perpendicular to the \( x \)-axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One-quarter of this square and one-eighth of \( S \) are shown. The area of this quarter-square is \( |PQ|^2 = r^2 - x^2 \). Therefore, \( A(x) = 4(r^2 - x^2) \) and the volume of \( S \) is
\[ V = \int_{-r}^{r} A(x) \, dx = 4 \int_{-r}^{r} (r^2 - x^2) \, dx \]
\[ = 8(r^2 - x^2) \bigg|_{0}^{r} = 8 \left[ r^2 x - \frac{1}{3} x^3 \right]_{0}^{r} = \frac{16}{3} r^3 \]