Section 11.3

32. (a) \( f(x) = \frac{1}{x^4} \) is positive and continuous and \( f'(x) = -\frac{4}{x^5} \) is negative for \( x > 0 \), and so the Integral Test applies.

\[
\sum_{n=1}^{\infty} \frac{1}{n^4} \approx s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \frac{1}{10^4} \approx 1.082037.
\]

\[
R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{3x^3} \right]_{10}^{t} = \lim_{t \to \infty} \left( -\frac{1}{3t^3} + \frac{1}{3(10)^3} \right) = \frac{1}{3000}, \text{ so the error is at most } 0.00033.
\]

(b) \( s_{10} + \int_{11}^{\infty} \frac{1}{x^4} \, dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^4} \, dx \Rightarrow s_{10} + \frac{1}{3(11)^3} \leq s \leq s_{10} + \frac{1}{3(10)^3} \Rightarrow 1.082037 + 0.000250 = 1.082287 \leq s \leq 1.082037 + 0.000333 = 1.082370, \text{ so we get } s \approx 1.08233 \text{ with error } \leq 0.00005.
\]

(c) \( R_n \leq \int_{n}^{\infty} \frac{1}{x^4} \, dx = \frac{1}{3n^3} \). So \( R_n < 0.00001 \Rightarrow \frac{1}{3n^3} < \frac{1}{10^6} \Rightarrow 3n^3 > 10^5 \Rightarrow n > \frac{\sqrt[3]{10^5}}{3} \approx 32.2, \text{ that is, for } n > 32.

33. (a) \( f(x) = \frac{1}{x^2} \) is positive and continuous and \( f'(x) = -\frac{2}{x^3} \) is negative for \( x > 0 \), and so the Integral Test applies.

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \approx s_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{10^2} \approx 1.549768.
\]

\[
R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_{10}^{t} = \lim_{t \to \infty} \left( -\frac{1}{t} + \frac{1}{10} \right) = \frac{1}{10}, \text{ so the error is at most } 0.1.
\]

(b) \( s_{10} + \int_{11}^{\infty} \frac{1}{x^2} \, dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^2} \, dx \Rightarrow s_{10} + \frac{1}{11} \leq s \leq s_{10} + \frac{1}{10} \Rightarrow 1.549768 + 0.090909 = 1.640677 \leq s \leq 1.549768 + 0.1 = 1.649768, \text{ so we get } s \approx 1.64522 \text{ (the average of the 1.640677 and 1.649768) with error } \leq 0.005 \text{ (the maximum of 1.649768 – 1.64522 and 1.64522 – 1.640677, rounded up).}
\]

(c) \( R_n \leq \int_{n}^{\infty} \frac{1}{x^2} \, dx = \frac{1}{n} \). So \( R_n < 0.001 \) if \( \frac{1}{n} < \frac{1}{1000} \) \( \Leftrightarrow n > 1000 \).

36. \( f(x) = \frac{1}{x(\ln x)^2} \) is positive and continuous and \( f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3} \) is negative for \( x > 1 \), so the Integral Test applies.

Using (2), we need \( 0.01 > \int_{n}^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \left[ -\frac{1}{\ln x} \right]_{n}^{t} = \frac{1}{\ln n} \). This is true for \( n > e^{100} \), so we would have to take this many terms, which would be problematic because \( e^{100} \approx 2.7 \times 10^{43} \).

Section 11.5

2. \( \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2} \). Here \( a_n = (-1)^n \frac{n}{n+2} \). Since \( \lim_{n \to \infty} a_n \neq 0 \) (in fact the limit does not exist), the series diverges by the Test for Divergence.

4. \( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n+1}}. \) Now \( b_n = \frac{1}{\sqrt{n+1}} > 0, \) \( \{b_n\} \) is decreasing, and

\[ \lim_{n \to \infty} b_n = 0, \text{ so the series converges by the Alternating Series Test.} \]

6. \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\ln(n+4)} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n. \) Now \( b_n = \frac{1}{\ln(n+4)} > 0, \) \( \{b_n\} \) is decreasing, and \( \lim_{n \to \infty} b_n = 0, \) so

the series converges by the Alternating Series Test.
11. \( b_n = \frac{n^2}{n^3 + 4} > 0 \) for \( n \geq 1 \). \( \{b_n\} \) is decreasing for \( n \geq 2 \) since
\[
\left( \frac{x^2}{x^3 + 4} \right)' = \frac{(x^3 + 4)2x - x^2(3x^2)}{(x^3 + 4)^2} = \frac{x(2x^3 + 8 - 3x^3)}{(x^3 + 4)^2} = \frac{x(8 - x^3)}{(x^3 + 4)^2} < 0 \text{ for } x > 2.
\]
Also,
\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1/n}{1 + 4/n^3} = 0.
\]
Thus, the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 4} \) converges by the Alternating Series Test.

14. \( \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{\ln n}{n} \right) = 0 + \sum_{n=2}^{\infty} (-1)^{n-1} \left( \frac{\ln n}{n} \right) \). \( b_n = \frac{\ln n}{n} > 0 \) for \( n \geq 2 \), and if \( f(x) = \frac{\ln x}{x} \), then
\[
f'(x) = \frac{1 - \ln x}{x^2} < 0 \text{ for } x > e, \text{ so } \{b_n\} \text{ is eventually decreasing. Also,}
\]
\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\ln n}{n} = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0,
\]
so the series converges by the Alternating Series Test.

15. \( \sum_{n=1}^{\infty} \frac{\cos n \pi}{n^{3/4}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{3/4}} \). \( b_n = \frac{1}{n^{3/4}} \) is decreasing and positive and \( \lim_{n \to \infty} \frac{1}{n^{3/4}} = 0 \), so the series converges by the Alternating Series Test.

20. \( \sum_{n=1}^{\infty} \left( \frac{n}{5} \right)^n \) diverges by the Test for Divergence since \( \lim_{n \to \infty} \left( \frac{n}{5} \right)^n = \infty \Rightarrow \lim_{n \to \infty} \left( -\frac{n}{5} \right)^n \) does not exist.

23. The series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \) satisfies (i) of the Alternating Series Test because \( \frac{1}{(n+1)^6} < \frac{1}{n^6} \) and (ii) \( \lim_{n \to \infty} \frac{1}{n^6} = 0 \), so the series is convergent. Now \( b_2 = \frac{1}{5^2} = 0.000064 > 0.00005 \) and \( b_6 = \frac{1}{6^6} \approx 0.000002 < 0.00005 \), so by the Alternating Series Estimation Theorem, \( n = 5 \). (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

24. The series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5} \) satisfies (i) of the Alternating Series Test because \( \frac{1}{(n+1)^5} < \frac{1}{n^5} \) and (ii) \( \lim_{n \to \infty} \frac{1}{n^5} = 0 \), so the series is convergent. Now \( b_4 = \frac{1}{4^5} = 0.0004 > 0.0001 \) and \( b_6 = \frac{1}{5^5} = 0.000064 < 0.0001 \), so by the Alternating Series Estimation Theorem, \( n = 4 \). (That is, since the 5th term is less than the desired error, we need to add the first 4 terms to get the sum to the desired accuracy.)

32. If \( p > 0 \), \( \frac{1}{(n+1)^p} \leq \frac{1}{n^p} \) (\( \{1/n^p\} \) is decreasing) and \( \lim_{n \to \infty} \frac{1}{n^p} = 0 \), so the series converges by the Alternating Series Test.

If \( p \leq 0 \), \( \lim_{n \to \infty} \frac{(-1)^{n-1}}{n^p} \) does not exist, so the series diverges by the Test for Divergence. Thus, \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \) converges \( \iff \) \( p > 0 \).

Section 11.6

1. (a) Since \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 8 > 1 \), part (b) of the Ratio Test tells us that the series \( \sum a_n \) is divergent.

(b) Since \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8 < 1 \), part (a) of the Ratio Test tells us that the series \( \sum a_n \) is absolutely convergent (and therefore convergent).

(c) Since \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), the Ratio Test fails and the series \( \sum a_n \) might converge or it might diverge.
4. \[ \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \frac{n}{n^4} \] diverges by the Test for Divergence. \[ \lim_{n \to \infty} \frac{2^n}{n^3} = \infty, \] so \[ \lim_{n \to \infty} \left( -1 \right)^{n+1} \frac{n}{n^4} \] does not exist.

5. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \] converges by the Alternating Series Test, but \[ \sum_{n=1}^{\infty} \frac{1}{n} \] is a divergent p-series \([p = \frac{1}{2} \leq 1]\), so the given series is conditionally convergent.

6. \[ \sum_{n=1}^{\infty} \frac{1}{n^4} \] is a convergent p-series \([p = 4 > 1]\), so \[ \sum_{n=1}^{\infty} \left( \frac{1}{n^4} \right) \] is absolutely convergent.

7. \[ \lim_{n \to \infty} \frac{\frac{1 + 1}{n^3}}{\frac{100}{n^4}} = \frac{\frac{n + 1}{100}}{n!} = \frac{n + 1}{100n!} \] is divergent by the Ratio Test.

8. \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{n + 1}{100n^3}}{100} = \frac{n + 1}{n^3} \] is divergent by the Ratio Test.

9. \[ \sum_{n=1}^{\infty} \frac{2^n}{n^3} \] converges by the Alternating Series Test since \[ \lim_{n \to \infty} \frac{1}{\ln n} = 0 \] and \[ \frac{1}{\ln n} \] is decreasing. Now \( \ln n < n \), so \[ \frac{\frac{1}{\ln n}}{n} \] converges by the Comparison Test. Thus, \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \] is conditionally convergent.

10. \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n + 1)!}{(n + 1)^{n+1} n!} = \lim_{n \to \infty} \frac{n^{n+1}}{(n + 1)^n} = \lim_{n \to \infty} \frac{1}{(1 + 1/n)^n} = \frac{1}{e} < 1, \] so the series \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \] converges absolutely by the Ratio Test.

11. \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \] is a convergent p-series \([p = 3 > 1]\), \[ \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3} \] converges, and so \[ \sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3} \] is absolutely convergent.

12. \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n + 1)!}{(n + 1)^{n+1} n!} = \lim_{n \to \infty} \frac{n^{n+1}}{(n + 1)^n} = \lim_{n \to \infty} \frac{1}{(1 + 1/n)^n} = \frac{1}{e} < 1, \] so the series \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \] converges absolutely by the Ratio Test.

13. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \] is absolutely convergent.

14. \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \] is a convergent p-series \([p = 3 > 1]\), \[ \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3} \] converges, and so \[ \sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3} \] is absolutely convergent.

15. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \] is absolutely convergent.

16. \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \] is a convergent p-series \([p = 3 > 1]\), \[ \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3} \] converges, and so \[ \sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3} \] is absolutely convergent.

17. \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n + 1)!}{(n + 1)^{n+1} n!} = \lim_{n \to \infty} \frac{n^{n+1}}{(n + 1)^n} = \lim_{n \to \infty} \frac{1}{(1 + 1/n)^n} = \frac{1}{e} < 1, \] so the series \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \] converges absolutely by the Ratio Test.

18. \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n + 1)!}{(n + 1)^{n+1} n!} = \lim_{n \to \infty} \frac{n^{n+1}}{(n + 1)^n} = \lim_{n \to \infty} \frac{1}{(1 + 1/n)^n} = \frac{1}{e} < 1, \] so the series \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \] converges absolutely by the Ratio Test.

19. \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n + 1)!}{(n + 1)^{n+1} n!} = \lim_{n \to \infty} \frac{n^{n+1}}{(n + 1)^n} = \lim_{n \to \infty} \frac{1}{(1 + 1/n)^n} = \frac{1}{e} < 1, \] so the series \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \] converges absolutely by the Ratio Test.

20. \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^{n+1} (n + 1)!}{5 \cdot 8 \cdot 11 \cdots (3n + 5)} = \lim_{n \to \infty} \frac{2^{n+1} (n + 1)!}{5 \cdot 8 \cdot 11 \cdots (3n + 2)} = \lim_{n \to \infty} \frac{2(n + 1)}{3n + 5} = \frac{2}{3} < 1, \] so the series converges absolutely by the Ratio Test.

21. (a) \[ \lim_{n \to \infty} \frac{1}{1/n^3} = \lim_{n \to \infty} \frac{n^3}{(n + 1)^3} = \lim_{n \to \infty} \frac{1}{(1 + 1/n)^3} = 1. \] Inconclusive

(b) \[ \lim_{n \to \infty} \frac{(n + 1)^2}{2n^3} = \lim_{n \to \infty} \frac{n + 1}{2n} = \lim_{n \to \infty} \frac{1/2 + 1/2n}{1} = \frac{1}{2}. \] Conclusive (convergent)

(c) \[ \lim_{n \to \infty} \frac{(n + 1)^2}{(3n + 5)^3} = \lim_{n \to \infty} \frac{1}{3n + 5} = \frac{1}{3}. \] Conclusive (convergent)

(d) \[ \lim_{n \to \infty} \frac{1/(n + 1)^2}{1/3n^2} = \lim_{n \to \infty} \frac{1}{1/n^2 + (1 + 1/n)^3} = 1. \] Inconclusive

30. By the recursive definition, \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2 + \cos n}{\sqrt{n}} = 0 < 1, \] so the series converges absolutely by the Ratio Test.

31. (a) \[ \lim_{n \to \infty} \frac{1}{(n + 1)^3} \] is inconclusive.

(b) \[ \lim_{n \to \infty} \frac{(n + 1)^2}{2n^3} = \lim_{n \to \infty} \frac{n + 1}{2n} = \lim_{n \to \infty} \frac{1/2 + 1/2n}{1} = \frac{1}{2}. \] Conclusive (convergent)

(c) \[ \lim_{n \to \infty} \frac{(-3)^n}{\sqrt{n + 1}} \] is inconclusive.

(d) \[ \lim_{n \to \infty} \frac{\sqrt{n + 1}}{1 + (n + 1)^2} \] is inconclusive.