Taylor Polynomials

The linear approximation to \( f \) at \( a \) is

\[
L(x) = f(a) + f'(a)(x-a)
\]

It's the 1st Order Taylor polynomial of \( f \) at \( a \)

\[
p_1(x) = L(x) = f(a) + f'(a)(x-a)
\]

Notice
- \( p_1(a) = f(a) \)
- \( p_1'(a) = f'(a) \)
  
  (because \( p_1'(x) = 0 + f'(a) \frac{d}{dx}(x-a) \)
  
  \( = f'(a) \cdot 1 \)

2nd Order Taylor Polynomial (Quadratic Approximation)

Idea! Approximate \( f \) by a degree 2 polynomial with

- \( p_2(a) = f(a) \)
- \( p_2'(a) = f'(a) \)
- \( p_2''(a) = f''(a) \)

The 2nd Order Taylor Polynomial of \( f \) at \( a \) is

\[
p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2
\]

It's also called the quadratic approximation of \( f \) at \( a \).

\[
p_2(a) = f(a)
\]
\[
p_2'(x) = f'(a) + \frac{1}{2}f''(a) \cdot 2(x-a) = f'(a) + f''(a)(x-a)
\]
\[
p_2'(a) = f'(a)
\]
\[
p_2''(x) = f''(a)
\]
\[
p_2''(a) = f''(a)
\]
**nth Order Taylor Polynomial**

The $n$th order Taylor polynomial of $f$ at $a$ is the degree $n$ polynomial that satisfies:

- $p_n(a) = f(a)$
- $p_n'(a) = f'(a)$
- $p_n''(a) = f''(a)$
- $\vdots$
- $p_n^{(n)}(a) = f^{(n)}(a)$

If $p_n$ and $f$ match at $x = a$.

\[
p_n(x) = \frac{1}{0!} f(a) + \frac{1}{1!} f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n
\]

- $0! = 1$
- $1! = 1$
- $2! = 1 \cdot 2 = 2$
- $3! = 1 \cdot 2 \cdot 3 = 6$
- $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$
- \vdots
- $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$

\[
p_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k
\]

**Exercise**

Verify:

- $p_1(a) = f(a)$
- $p_2(a) = f'(a)$
- $p_3(a) = f''(a)$
- $p_4(a) = f^{(3)}(a)$
- $p_5(a) = f^{(4)}(a)$
Example

Find the 3rd order Taylor polynomial of \( f(x) = \sqrt{x} \) at \( a = 4 \).

Use it to approximate \( \sqrt{3.98} \).

\[
\begin{align*}
 f(x) &= \sqrt{x} \\
 f'(x) &= \frac{1}{2} x^{-1/2} \\
 f''(x) &= -\frac{1}{4} x^{-3/2} \\
 f'''(x) &= \frac{3}{8} x^{-5/2} \\
 f(4) &= 2 \\
 f'(4) &= \frac{1}{2} \cdot 1 \cdot (4)^{-1/2} = \frac{1}{2} \cdot 4^{-1/2} = \frac{1}{4} \\
 f''(4) &= -\frac{1}{4} \cdot (4)^{-3/2} = -\frac{1}{4} \cdot 2^{-3} = -\frac{1}{32} \\
 f'''(4) &= \frac{3}{8} \cdot (4)^{-5/2} = \frac{3}{8} \cdot \frac{1}{2^5} = \frac{3}{256}
\end{align*}
\]

\[
\begin{align*}
p_3(x) &= f(4) + f'(4)(x - 4) + \frac{f''(4)}{2!}(x - 4)^2 + \frac{f'''(4)}{3!}(x - 4)^3 \\
&= 2 + \frac{1}{4} (x - 4) + \frac{1}{2} \left( -\frac{1}{32} \right) (x - 4)^2 + \frac{1}{6} \left( \frac{3}{256} \right) (x - 4)^3
\end{align*}
\]

\[
\sqrt{3.98} = f(3.98) \approx p_3(3.98)
\]

\[
= 2 + \frac{1}{4} (3.98 - 4) + \frac{1}{2} \left( -\frac{1}{32} \right) (3.98 - 4)^2
\]

\[
+ \frac{1}{6} \left( \frac{3}{256} \right) (3.98 - 4)^3
\]

\[
= 1.994993734375\ldots
\]

Linear approximation at \( a = 4 \):

\[
\sqrt{3.98} = f(3.98) \approx L(3.98) = 1.995
\]

The exact value of \( \sqrt{3.98} \) is

\[
\sqrt{3.98} = 1.994993734375\ldots
\]