Price Elasticity of Demand

\[ e = e(p) = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{\text{percentage change in demand}}{\text{percentage change in price}} \]

Interpretation.

Law of Demand: Demand decreases as price increases (and vice versa)

\[ \frac{dq}{dp} < 0 \]

Since \( p, q \) are always \( \geq 0 \),

\[ e < 0 \]

\[ \frac{dR}{dp} = \frac{d}{dp} (pq) = q + p \frac{dq}{dp} = q \left[ 1 + \frac{p}{q} \cdot \frac{dq}{dp} \right] = q \left[ 1 + e \right] \]

\[ \frac{dR}{dp} = q \left[ 1 + e \right] \]

Note \( q > 0 \).

- \( |e| > 1 \)
- \( e < -1 \)
- \( \frac{dR}{dp} < 0 \)
- Revenue decreases as price increases (and vice versa)

\[ \% \Delta q > \% \Delta p \]

\[ e = \frac{\% \Delta q}{\% \Delta p} \]
1. change in price implies a greater than 1% change in demand
2. demand is price elastic (price sensitive)

- \( |\varepsilon| < 1 \)
- \(-1 < \varepsilon < 0 \)

- \( \frac{dR}{dp} > 0 \)  \hspace{1cm} \frac{dR}{dp} = q|1+\varepsilon| \)
- revenue increases as price increases (and vice versa)

- \( \frac{\gamma \Delta q}{\gamma \Delta p} < \frac{\Delta p}{\Delta q} \)
- percentage change in demand < percentage change in price

- 1% change in price implies a less than 1% change in demand
- demand is price inelastic (price insensitive)

- \( |\varepsilon| = 1 \)
- \( \varepsilon = -1 \)

- \( \frac{dR}{dp} = 0 \)  \hspace{1cm} \frac{dR}{dp} = q|1+\varepsilon| \) \( q > 0 \)
- revenue does not change as price changes

- \( \frac{\gamma \Delta q}{\gamma \Delta p} = \frac{\Delta p}{\Delta q} \)
- percentage change in demand = percentage change in price
1. A change in price implies a 1% change in demand.
2. Demand is unit elastic when revenue is maximized,
   \[
   \frac{dR}{dp} = 0
   \]
   which means
   \[
   E = -1 \quad \text{(unit elasticity)}
   \]

**Example**

Suppose the demand curve for oPads is given by
\[
y = 500 - 10p
\]

(a) Compute the price elasticity of demand.
\[
\frac{dy}{dp} = -10
\]
\[
E(p) = \frac{p}{y} \cdot \frac{dy}{dp} = \frac{p}{500-10p} \cdot (-10) = \frac{-10p}{p-50}
\]

(b) What is the price elasticity of demand when the price is $30?
\[
E(30) = \frac{30}{30-80} = \frac{30}{-50} = -\frac{3}{2} = -1.5
\]

(c) What is the percent change in demand if the price is $30 and increases by 4.5%?
\[
E = \frac{\gamma \cdot \Delta y}{\gamma \cdot \Delta p}
\]
   decreases
\[ Y \Delta q = e^0(Y - \Delta p) \]
\[ Y \Delta q \bigg|_{p=30} = e(30) \cdot (0.045) = -\frac{3}{2} \cdot (0.045) \]
\[ -0.0675 = -0.0675 \]
\[ = -6.7875 \%
\]

(d) What should management do if the price is $30?

\[ e(30) = -\frac{3}{2} < -1 \]

The demand for the good is price elastic, so management should decrease the price.

\[ \frac{dR}{dp} = q(1 + eI) < 0 \]

Decrease in price will increase revenue.
Continuous Compound Interest

Compound Interest

\[ A = P \left(1 + \frac{r}{n}\right)^{nt} \]

- \( P \) = principal (initial) amount
- \( r \) = annual interest rate
- \( n \) = number of compound periods per year
- \( t \) = number of years.
- \( A \) = amount after \( t \) years.

Example: If you invest $1,000 at a 10% annual interest rate compounded twice per year for 3 years, you end up with

\[ A = 1000 \left(1 + \frac{0.10}{2}\right)^{2 \cdot 3} \]

\[ = 1000 \left(1.05\right)^6 \]

\[ \approx \$1348.01 \]

Continuous Compound Interest

If we compound continuously, then \( n \) (the number of compound periods per year) goes to \( \infty \)

\[ A = \lim_{n \to \infty} P \left(1 + \frac{r}{n}\right)^{nt} \]

\[ = \lim_{n \to \infty} P \left[ \left(1 + \frac{r}{n}\right)^n\right]^t \]

\[ = P \lim_{n \to \infty} \left[ \left(1 + \frac{r}{n}\right)^n\right]^t \]
\[ = P \left[ \lim_{n \to \infty} \left( 1 + \frac{r}{n} \right)^n \right]^+ \]

Fact: \[ \lim_{n \to \infty} \left( 1 + \frac{r}{n} \right)^n = e^r \]

\[ A = Pe^{rt} \]

Continuous Compound Interest Formula

Example

If we compounded continuously in the example above, we would end up with

\[ A = 10000 \left( e^{0.10(3)} \right) \approx 13498.66 \]

We can use the continuous formula as an easier-to-use approximation to the discrete formula.

Example

Suppose your parents want to buy you a $80,000 car as a graduation gift in 4 years. Assume they can invest at an 8% interest rate compounded continuously.

How much should they invest now to get you that car when you graduate?

\[ A = Pe^{rt} \]

Find P.

\[ Ae^{-rt} = P \]

\[ P = Ae^{rt} = (80000) \cdot e^{-0.08 \cdot 4} \approx 36307.45 \]
\[ P = \frac{A}{e^{rt}} = \frac{5000}{e^{0.08}(4)} \approx 36307.45 \]

**Example**

Consider an investment that earns an annual interest rate of 6\% compounded continuously.

(a) How fast is the investment growing when its value is $5000?

(b) If the initial amount invested is $3000, how many years are required for the value of the investment to reach $4000.

(a) \[ A = Pe^{rt} \]

\[ \frac{dA}{dt} = A Pe^{rt} = Pe^{rt} \cdot r = Ar \]

When the amount is $5000, the growth rate is

\[ \frac{dA}{dt} = (5000)(0.06) \quad \text{\$ per year} \]

(b) \[ A = Pe^{rt} \]

\[ \frac{A}{P} = e^{rt} \]

\[ \ln\left(\frac{A}{P}\right) = rt \]

\[ t = \ln\left(\frac{A}{P}\right) \cdot \frac{1}{r} = \ln\left(\frac{4000}{3000}\right) \cdot \frac{1}{0.06} \quad \text{years} \]