Maxima and Minima (Extrema)

**Absolute Extrema**

$f(c)$ is an absolute maximum value of $f$ if $f(c) \geq f(x)$ for all $x$ in the domain of $f$.

$f(c)$ is an absolute minimum value of $f$ if $f(c) \leq f(x)$ for all $x$ in the domain of $f$.

**Absolute Extrema on a Set**

Suppose $D$ is a set of real numbers containing $c$.

$f(c)$ is an absolute maximum value of $f$ on $D$ if $f(c) \geq f(x)$ for all $x$ in $D$.

$f(c)$ is an absolute minimum value of $f$ on $D$ if $f(c) \leq f(x)$ for all $x$ in $D$.

**Absolute Extrema on $[3,5]$**

- $f(3)$ is the absolute maximum of $f$ on $[3,5]$
- $f(4)$ is the absolute minimum of $f$ on $[3,5]$

**Absolute Extrema on $(3,5)$**

- $f(4)$ is the absolute minimum of $f$ on $(3,5)$
- $f$ has no absolute maximum on $(3,5)$

**Absolute Extrema on $(3.1,4.8)$**

- $f(4)$ is absolute minimum of $f$ on $(3.1,4.8)$
- $f$ has no absolute maximum on $(3.1,4.8)$

**Extrema on the domain**

- $f(2)$ is the absolute maximum of $f$
- $f$ has no absolute minimum

Local Extrema

$f(2)$ and $f(6)$ are local maxima of $f$.

$f(4)$ is a local minimum of $f$.

$f(0)$ is not a local minimum of $f$. 

$y = f(x)$, domain $= [0, \infty)$

$\lim_{{x \to \infty}} f(x) = 0$
Local Extrema

\( f(c) \) is a local maximum of \( f \) if \( f(c) \geq f(x) \) for all \( x \) in some open interval around \( c \) (Think: for all \( x \) near \( c \)).

\( f(x) \) is a local minimum of \( f \) if \( f(c) \leq f(x) \) for all \( x \) in some open interval around \( c \) (Think: for all \( x \) near \( c \)).

Extra Example

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Remark: According to the definition, local maxima and local minima cannot occur at the endpoints of the domain.

Extreme Value Theorem

If \( f \) is continuous on the closed and bounded interval \([a,b]\), then \( f \) has an absolute maximum and an absolute minimum on \([a,b]\).
Observation

Unless it occurs at an endpoint, each absolute extremum of \( F \) on \([a,b]\) is a local extremum.

**Fermat’s Theorem**

If \( F \) has a local extremum at \( c \), then \( F'(c) = 0 \) or \( F'(c) \) doesn’t exist. \( c \) is a critical point of \( F \).

**Critical Points**

A point \( c \) in the domain of \( F \) where \( F'(c) = 0 \) or \( F'(c) \) doesn’t exist is called a critical point of \( F \).

**Locating Absolute Extremum Values on a Closed and Bounded Interval \([a, b]\)**

If \( F \) is continuous on the closed and bounded interval \([a, b]\), then the following procedure will determine the absolute extrema of \( F \) on \([a, b]\).

1. Find the critical points of \( F \) in \((a,b)\).
2. Evaluate \( F \) (not \( F' \)) at the critical points in \((a,b)\) and at the endpoints \( a \) and \( b \).
3. The largest value of \( F \) from Step 2 is the absolute maximum of \( F \) on \([a,b]\).
   The smallest value of \( F \) from Step 2 is the absolute minimum of \( F \) on \([a,b]\).
Example

Find the absolute maximum and minimum values of
\( f(x) = x^3 - 6x^2 + 1 \) on the interval \([-2, 1]\).

\( f'(x) = 3x^2 - 12x \)

\( f'(x) \) exists everywhere.

Set \( 0 = f'(x) \)

\[ 0 = 3x^2 - 12x \]
\[ 0 = x^2 - 4x \]
\[ 0 = x(x - 4) \]
\[ x = 0 \quad \text{or} \quad x = 4 \]

CPs in \((-2, 1)\): \( 0 \)

Evaluate \( f \) at CPs in \((-2, 1)\) and at the endpoints \(-2, 1\)

\[ f(0) = 0^3 - 6(0)^2 + 1 = 1 \] \( \text{abs. max} \)

\[ f(-2) = (-2)^3 - 6(-2)^2 + 1 = -8 - 24 + 1 = -31 \] \( \text{abs. min} \)

\[ f(1) = (1)^3 - 6(1)^2 + 1 = -4 \]

Therefore:

\[ f(0) = 1 \] is the absolute maximum of \( f \) on \([-2, 1]\)

\[ f(-2) = -31 \] is the absolute minimum of \( f \) on \([-2, 1]\)

Example

Find the absolute maximum and minimum values of
\( f(x) = x^{2/3} \) on \([-8, 8]\).

\( f'(x) = \frac{2}{3}x^{-\frac{1}{3}} \)

\( f'(x) \) does not exist at \( x = 0 \)

Set \( 0 = f'(x) = \frac{2}{3}x^{-\frac{1}{3}} \). No solution.
CPs in \((-8,8)\): 0

\[ f(0) = 0^{\frac{2}{3}} = 0 \]
\[ f(-8) = (-8)^{\frac{2}{3}} = (-2)^2 = 4 \]
\[ f(8) = (8)^{\frac{2}{3}} = 2^2 = 4 \]

\[ f(0) = 0 \text{ is the absolute min of } f \text{ on } [-8,8] \]
\[ f(-8) = f(8) = 4 \text{ is the absolute max of } f \text{ on } [-8,8] \]

**Extra Example**

Find the absolute maximum and minimum values of

\[ f(x) = x^2e^{-x} \text{ on } [-3,3]. \]

\[ f'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x} \]

**f(x) exists everywhere**

Set \( 0 = f'(x) \)

\( 0 = e^{-x}x(2-x) \)

\( x = 0 \text{ or } x = 2 \)

CPs in \((-3,3)\): 0, 2

\[ f(0) = 0^2e^0 = 0 \text{ ← smallest} \]
\[ f(2) = 2^2e^{-2} = 4e^{-2} \]
\[ f(-3) = (-3)^2e^{-(3)} = 9e^3 \text{ ← largest} \]
\[ f(3) = 3^2e^{-3} = 9e^{-3} \]

\( f(0) \text{ is the absolute minimum value of } f \text{ on } [-3,3] \)
\( f(-3) \text{ is the absolute maximum value of } f \text{ on } [-3,3]. \)