Derivatives as Rates of Change

One-Dimensional Motion

- An object moving in a straight line
- For an object moving in more complicated ways, consider the motion of the object in just one of the three dimensions in which it moves.

For an object moving in one-dimension, its position at time \( t \) is given by its position function \( s = f(t) \).

Average Velocity

Over the time interval \([t_0, t_1]\),

\[
\text{Elapsed Time} = \Delta t = t_1 - t_0
\]

\[
\text{Displacement} = \Delta s = f(t_1) - f(t_0)
\]

\[
\text{Average Velocity} = \frac{\Delta s}{\Delta t} = \frac{f(t_1) - f(t_0)}{t_1 - t_0} = \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}
\]

The average velocity is the slope of the secant line to the position curve \( s = f(t) \) between \((t_0, f(t_0))\) and \((t_1, f(t_1))\).

Velocity (Instantaneous Velocity)

The (instantaneous) velocity at time \( t_0 \) is

\[
v(t_0) = \frac{ds}{dt}(t_0) = f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}.
\]

The (instantaneous) velocity at time \( t_0 \) is the slope of the tangent line to the position curve \( s = f(t) \) at \((t_0, f(t_0))\).

The velocity function is

\[
v = \frac{ds}{dt} = f'.
\]

Acceleration

The acceleration at time \( t_0 \) is

\[
a(t_0) = \frac{dv}{dt}(t_0) = \frac{d^2s}{dt^2}(t_0) = f''(t_0).
\]

The acceleration function is

\[
a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''.
\]

Speed

Speed is the absolute value of the velocity,

\[
\text{speed} = |v|
\]
Example: A stone thrown vertically upwards... on Mars!!!

You are an astronaut at the edge of a cliff on Mars. You throw a stone straight up with an initial velocity of 64 ft/s from a height of 152 ft above the ground. The height $s$ of the stone above the ground after $t$ seconds is given by

$$s = -6t^2 + 64t + 152$$

(a) Determine the velocity of the stone after $t$ seconds.

(b) When does the stone reach its highest point?

(c) What is the height of the stone at the highest point?

(d) When does the stone strike the ground?

(e) With what velocity does the stone strike the ground?

(f) Determine the acceleration of the stone after $t$ seconds.

(g) What is this acceleration and why is it constant?

Solution.

(a) $v(t) = \frac{ds}{dt} = \frac{d}{dt}(-6t^2 + 64t + 152) = -12t + 64$

(b) When height $s(t)$ is maximized,

$$v(t) = \frac{ds}{dt} = 0$$

$$-12t + 64 = 0$$

$$t = \frac{64}{12} = \frac{16}{3} \text{ seconds}$$

(c) The maximum height is

$$s\left(\frac{64}{12}\right) = -6\left(\frac{64}{12}\right)^2 + 64\left(\frac{64}{12}\right) + 152 = \frac{968}{3} = 322.666\ldots \text{ ft.}$$

(d) The stone strikes the ground when

$$s(t) = 0$$

$$-6t^2 + 64t + 152 = 0$$

$$t = \frac{-64 \pm \sqrt{(64)^2 - 4(-6)(152)}}{2(-6)}$$

$$t = \frac{-64 \pm \sqrt{7744}}{2(-6)}$$

$$t = \frac{-64 \pm 88}{-12}$$
\[ t = \frac{-64 + 88}{-12} \quad \text{or} \quad t = \frac{-64 - 88}{-12} \]

\[ t = \frac{24}{-12} \quad \text{or} \quad t = \frac{-152}{-12} \]

\[ t = -2 \quad \text{or} \quad t = \frac{152}{12} \]

\[ t = \frac{152}{12} \] seconds

(e) The stone strikes the ground at velocity

\[ v \left(\frac{152}{12}\right) = -12 \left(\frac{152}{12}\right) + 64 = -88 \text{ ft/s} \]

(f) \[ a(t) = \frac{dv}{dt} = \frac{d}{dt}(-12t + 64) = -12 \text{ ft/s}^2 \]

(g) The acceleration is constant because only the constant force of gravity acts on the stone. The constant \(-12 \text{ ft/s}^2\) is the acceleration due to gravity on Mars.

For reference, the acceleration due to gravity on Earth is \(-32 \text{ ft/s}^2\), so you would feel almost three times lighter on Mars than on Earth. The mass of Mars is about one-tenth the mass of Earth, and the radius of Mars is about half the radius of Earth.
Other Rates of Change: An Example

In 2005, the world population was about 6.454 billion. In 2010, it was about 6.972 billion.

What is the average growth rate of the world population between 2005 and 2010?

The average growth rate of the world population between 2005 and 2010 is

$$\frac{\Delta p}{\Delta t} = \frac{6.972 \times 10^9 - 6.454 \times 10^9}{2010 - 2005} = \frac{(6.972 - 6.454) \cdot 10^9}{5} = 103,600,000 \text{ people/year}$$

If we fit an exponential curve to the data, we get a model that predicts the world population at year $t$ is

$$p(t) = 0.0002317e^{0.0154404t}$$

According to the model, what is the average growth rate of the world population between 1995 and 2015?

The average growth rate of the world population between 1995 and 2015 is

$$\frac{\Delta p}{\Delta t} = \frac{p(2015) - p(1995)}{2015 - 1995} = 100,045,668 \text{ people/year}$$

According to the model, what is the (instantaneous) growth rate in 2014? In 2020?

The growth rate at year $t$ is

$$p'(t) = \frac{dp}{dt} = \frac{d}{dt}0.0002317e^{0.0154404t} = (0.0002317)(0.0154404)e^{0.0154404t}$$

The growth rate in 2014 is

$$p'(2014) = (0.0002317)(0.0154404)e^{0.0154404(2014)} \approx 114.5 \text{ million people/year}$$

The growth rate in 2020 is

$$p'(2020) = (0.0002317)(0.0154404)e^{0.0154404(2020)} \approx 125.6 \text{ million people/year}$$

According to the model, what will the world population be in 2100?

In 2100, the world population will be

$$p(2100) = 0.0002317e^{0.0154404(2100)} \approx 27.98 \text{ billion}$$

Is this realistic? Why or why not?
Rates of Change in Business and Economics

**Basic Definitions**

\( p \) = unit price.

\( q \) = quantity demanded. It is the number of units that will sell at price \( p \).

We typically assume that the number of units produced is exactly the number of units that will sell. Therefore we also have:

\( q \) = quantity produced.

Sometimes we treat \( q \) as a function of \( p \). Other times we treat \( p \) as a function of \( q \).

An equation relating \( p \) and \( q \) is called a demand equation. The curve describing the relationship between \( p \) and \( q \) is called the demand curve.

**Profit** = \( P = R - C \)

**Revenue** = \( R = pq \).

Cost \( C \) is the sum of fixed cost \( FC \) and variable cost \( VC \), i.e.,

\[ C = FC + VC. \]

Variable costs \( VC \) and hence costs \( C \) will be a function of price \( p \) or \( q \), depending on which we regard as the independent variable.

**Average Cost and Marginal Cost**

The cost function \( C(q) \) gives the cost to produce the first \( q \) items.

The average cost to produce the first \( q \) items is

\[ \overline{C}(q) = \frac{C(q)}{q} \]

The marginal cost is

\[ MC(q) = C'(q) = \frac{dC}{dq} \]

The marginal cost is interpreted as the cost to produce one additional item after producing \( q \) items.

**Optional Note**

Technically, the cost to produce one additional item after producing \( q \) items is

\[ C(q + 1) - C(q) = \frac{C(q + 1) - C(q)}{q + 1 - q} = \frac{\Delta C}{\Delta q}. \]

But when \( q \) is large or the slope of the cost curve is nearly constant near \( q \), we have

\[ \frac{\Delta C}{\Delta q} \approx \frac{dC}{dq}. \]

This is usually the case in practice.
Example.
Suppose the cost of producing \( q \) items is given by the function
\[
C(q) = -q^2 + 500q + 1000
\]

(a) Determine the fixed cost.
(b) Determine the variable cost.
(c) Determine the average cost and marginal cost functions.
(d) Determine the average cost for the first 200 items.
(e) Determine the marginal cost for the first 200 items.

Solution.
(a) \( FC = 1000 \)
(b) \( VC(q) = -q^2 + 500q \)
(c) \( \overline{C}(q) = \frac{-q^2 + 500q + 1000}{q} = -q + 500 + \frac{1000}{q} \)
\( C'(q) = -2q + 500 \)
(d) \( \overline{C}(200) = \frac{C(200)}{200} = \frac{-(200)^2 + 500(200) + 1000}{200} = \frac{-40000 + 100000 + 1000}{200} \)
\[= \frac{61000}{200} = 305 \text{ \$/unit} \]
(e) \( C'(200) = -2(200) + 500 = 100 \text{ \$/unit} \)

Average Revenue and Marginal Revenue
The revenue function \( R(q) = pq \) gives the revenue for selling the first \( q \) items. We are regarding \( p \) as a function of \( q \) here.

The average revenue for selling the first \( q \) items is just the unit price:
\[
\overline{R}(q) = \frac{R(q)}{q} = \frac{pq}{q} = p
\]

The marginal revenue is
\[
MR(q) = R'(q) = \frac{dR}{dq}
\]

The marginal revenue is interpreted as the revenue for selling one additional item after selling \( q \) items.

Average and Marginal Profit
The profit function \( P(q) = R(q) - C(q) \) gives the profit for the first \( q \) items.
The average profit for the first \( q \) items is
\[
\bar{P}(q) = \frac{P(q)}{q}
\]
The marginal profit is
\[
MP(q) = P'(q) = \frac{dP}{dq}
\]
The marginal profit is interpreted as the profit for one additional item after producing and selling \( q \) items.

**Break-Even Points**

A number \( q \) is called a break-even point if \( P(q) = 0 \), in other words, if \( R(q) = C(q) \).

**Example.**

Westeros Inc. manufactures Valyrian steel swords. When the price is $3000, 300 swords are sold. For every $100 increase in price, 20 fewer swords are sold. It costs \( C(q) = -4q^2 + 3500q + 50000 \) dollars to produce the first \( q \) swords.

(a) Determine the linear demand equation.

(b) Determine the revenue function.

(c) Determine the profit function.

(d) Suppose 800 swords are produced. If one additional sword is produced and sold, does the profit increase or decrease?

(e) Show that there is a break-even point between \( q = 750 \) and \( q = 1000 \).

**Solution.**

(a) \( q - q_0 = m(p - p_0) \), where \( (p_0, q_0) \) is a point on the demand curve and \( m = \frac{\Delta q}{\Delta p} \)

\[
(p_0, q_0) = (3000, 300)
\]
\[
m = \frac{\Delta q}{\Delta p} = -\frac{20}{100} = -\frac{1}{5}
\]
\[
q - 500 = -\frac{1}{5}(p - 3000)
\]

(b) \( R(q) = pq \)

Rearrange the demand equation:
\[
-5(q - 300) = p - 3000
\]
\[
p = -5q + 1500 + 3000
\]
\[
p = -5q + 4500
\]
Therefore \( R(q) = (-5q + 4500)q = -5q^2 + 4500q \)
(c) $P(q) = R(q) - C(q) = -5q^2 + 4500q - (-4q^2 + 3500q + 50000) = -q^2 + 1000q - 50000$

(d) The marginal profit is the profit for one additional sword after producing and selling $q$ swords.

$$P'(q) = -2q + 1000$$

$$P'(800) = -2(800) + 1000 < 0$$

Therefore the profit decreases if one additional sword is produced and sold after producing and selling 800 swords.

(e) At a break-even point

$$P(q) = -q^2 + 1000q - 50000 = 0$$

$$P(1000) = -(1000)^2 + 1000(1000) - 50000 = -50000 < 0$$

$$P(750) = -(750)^2 + 1000(750) - 50000$$

$$= 750(-750 + 1000) - 50000$$

$$= 750(250) - 50000$$

$$> 700(200) - 50000$$

$$= 140000 - 50000 > 0$$

$P(q)$ is continuous on $[750, 1000]$

Therefore, by the IVT, there exists a $c$ in $(750, 1000)$ such that $P(c) = 0$. 