Inverse, Exponential, and Logarithmic Functions

Rules of Exponents

Example.

\[
\frac{2^3}{2^5} = 2^{3-5} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}
\]

The real number \( b > 0 \) is the base. \( x, y \) are any real numbers.

1. \( b^x b^y = b^{x+y} \)
2. \( \frac{1}{b^y} = b^{-y} \) and \( \frac{b^x}{b^y} = b^{x-y} \)
3. \( (b^x)^y = b^{xy} \)
4. \( b^x > 0 \)

Exponential Functions

\[ f(x) = b^x \quad \text{with base } b > 0 \]

\[
\begin{align*}
  y & = \left(\frac{1}{5}\right)^x \\
  y & = 5^x \\
  y & = \left(\frac{1}{3}\right)^x \\
  y & = 3^x
\end{align*}
\]
Properties of $f(x) = b^x$

1. Domain $= (-\infty, \infty) = \mathbb{R}$ = all real numbers.
2. Range $= (0, \infty)$ = all positive real numbers.
3. If $b > 1$, then $f$ is increasing ($f(x) < f(y)$ when $x < y$). 
4. If $b < 1$, then $f$ is decreasing ($f(x) > f(y)$ when $x < y$).
5. $f(0) = b^0 = 1$.

Reminder:

Domain = set of $x$ where $f(x)$ is defined.
Range = set of all output values of $f$.

The Natural Exponential Function

$$f(x) = e^x$$

The base is $e = 2.718281828459\ldots$

Inverse Functions

Given a function $f$, the inverse of $f$ is the function $f^{-1}$ that satisfies

$$f^{-1}(f(x)) = x$$

for all $x$ in the domain of $f$.

The inverse of $f$ may not exist.

One-to-One (1-1) Functions

A function is a rule that assigns one output to each input.

A function is called 1-1 if each possible output corresponds to exactly one input; in other words, the same output is never obtained from two different inputs.

$$f(x) = x^2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad f(x) = 2x$$

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$x^2$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>$-2$</td>
<td>4</td>
</tr>
</tbody>
</table>

$$f(x) = x^2$$ is not 1-1 because $f(2) = 4$ and $f(-2) = 4$.

$$f(x) = 2x$$ is 1-1. This is easily seen graphically with the Horizontal Line Test.
Horizontal Line Test

$f$ is 1-1 if and only if every horizontal line intersects the graph of $f$ at most once.

One-to-One on a Domain

A function $f$ is 1-1 on a domain $D$ if each value of $f(x)$ corresponds to exactly one value of $x$ in $D$.

Example.

$f(x) = x^2$ is not 1-1 on $(-\infty, \infty)$. $f(x) = x^2$ is 1-1 on $[0, \infty)$.
One-to-One Implies Inverse Exists

If $f$ is 1-1 on a domain $D$ with range $R$, then $f$ has an inverse on $D$, the inverse $f^{-1}$ has domain $R$ and range $D$, and

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y \quad (\text{for all } x \text{ in } D, \text{ for all } y \text{ in } R)$$

Example.

Above is the graph of $f(x) = x^3 - 3x$. Use it to determine the intervals where $f$ has an inverse.

The Horizontal Line Test shows $f$ is 1-1 on $(-\infty, -1], [-1, 1], [1, \infty)$.

\[\therefore f \text{ has an inverse on } (-\infty, -1], [-1, 1], [1, \infty)\]
Graphing Inverse Functions

The graphs of $f$ and $f^{-1}$ are symmetric about $y = x$ because if $(a, b)$ is on the graph of $f$, then $(b, a)$ is on the graph of $f^{-1}$.

Example.
Plot $f(x) = x^2 + 2$ ($x \geq 0$) and its inverse on the same graph.
Finding Inverse Functions

Example.

a. Find the inverse of \( f(x) = x^2 + 2 \) on the interval \( x \geq 0 \).

b. Verify \( f^{-1}(f(x)) = x \) and \( f(f^{-1}(x)) = x \).

\begin{align*}
a. \quad y &= x^2 + 2, \quad x \geq 0 \\
& \text{Solve for } x \text{ in terms of } y \\
x^2 &= y - 2 \\
x &= \pm \sqrt{y - 2} \\
& \quad \text{Since } x \geq 0, \quad x = \sqrt{y - 2} \\
& \quad \text{Switch } x \text{ and } y \\
y &= \sqrt{x - 2} \\
f^{-1}(x) &= \sqrt{x - 2} \\
& \text{b. } f^{-1}(f(x)) = f^{-1}(x^2 + 2) \\
& = \sqrt{(x^2 + 2) - 2} \\
& = \sqrt{x^2} \\
& = |x| \\
& = x \\
& \quad \text{Note } |x| = x \text{ because } x \geq 0 \\
f(f^{-1}(x)) &= f(\sqrt{x - 2}) \\
& = (\sqrt{x - 2})^2 + 2 \\
& = x - 2 + 2 \\
& = x
\end{align*}
Logarithmic Functions

The base $b$ logarithm function $\log_b x$ is the inverse of the exponential function $b^x$.

The natural logarithm function is $\ln x = \log_e x$. It is the inverse of the natural exponential function $e^x$.

**Inverse Relationship between $b^x$ and $\log_b x$**

1. $\log_b b^x = x$ for all real numbers $x$.
2. $b^{\log_b y} = y$ for all $y > 0$.
3. $x = \log_b y$ if and only if $b^x = y$.

**Properties of $\log_b x$**

1. Domain = $(0, \infty) = \text{all positive real numbers}$.
2. Range $= (-\infty, \infty) = \mathbb{R} = \text{all real numbers}$.

Note: The range of $b^x$ is the domain of $\log_b x$ (and vice versa) because they are inverses.

3. If $b > 1$, then $\log_b x$ is increasing (just like $b^x$).
4. If $b < 1$, then $\log_b x$ is decreasing (just like $b^x$).
5. $\log_b(1) = 0$ (because $b^0 = 1$).
Logarithm Rules

Example.
\[
\ln \left( \frac{2}{5} \right) = \ln (2 \cdot 5^{-1}) = \ln (2) + \ln (5^{-1}) = \ln 2 - \ln 5
\]

1. \( \log_b(xy) = \log_b x + \log_b y \)
2. \( \log_b \left( \frac{1}{y} \right) = \log_b(y^{-1}) = - \log_b y \)
3. \( \log_b \left( \frac{x}{y} \right) = \log_b(xy^{-1}) = \log_b x - \log_b y \)
4. \( \log_b x^z = z \log_b x \)
5. \( \log_b b = 1 \)

Change of Base

1. \( b^x = e^{x \ln b} \) for all \( x \)
   
   \[ [b^x = e^{\ln b^x} = e^{x \ln b}] \]
2. \( \log_b x = \frac{\ln x}{\ln b} \) for all \( x > 0 \)

Example.

Solve \( \log_5 x = -2 \).

Exponentiate both sides

\[
5^{\log_5 x} = 5^{-2}
\]

\[
x = 5^{-2}
\]

Example.

Solve \( 7^{2x+1} = 29 \).

Take log of both sides

\[
\log_7(7^{2x+1}) = \log_7(29)
\]

\[
2x + 1 = \log_7(29)
\]

\[
2x = \log_7(29) - 1
\]

\[
x = \frac{1}{2} \log_7(29) - \frac{1}{2}
\]