1. Find the plane through \((2, 1, 0)\) which is parallel to the plane \(x + 4y - 3z = 1 - 2y\).

In standard form, the given plane is \(x + 6y - 3z = 1\). So the plane we want has the form

\[ x + 6y - 3z = d. \]

Plugging in \((2, 1, 0)\) gives

\[ 2 + 6(1) - 3(0) = d, \]

so \(d = 8\). Therefore the plane we want is

\[ x + 6y - 3z = 8. \]

2. What type of curves are the \(y = k\) traces of \(4x^2 + y^2 - 2z = 0\)?

The \(y = k\) trace is

\[ 4x^2 + k^2 - 2z = 0 \]

\[ z = \frac{1}{2} (4x^2 + k^2) \]

\[ z = 2x^2 + \frac{1}{2} k^2 \]

So the \(y = k\) traces are parabolas.

4. The function \(f(x, y) = e^{2xy}\) has exactly one local extremum subject to \(x^3 + y^3 = 16\). Use Lagrange multipliers to find it.

\[
\begin{align*}
f(x, y) &= e^{2xy} \\
g(x, y) &= x^3 + y^3 \\
f_x &= 2y \\
g_y &= 3y^2 \\
f_y &= 2xy \\
g_x &= 3x^2
\end{align*}
\]

We need to solve the system of equations

\[
\begin{align*}
(1) & \quad e^{2xy} \cdot 2y = \lambda \cdot 3x^2 \\
(2) & \quad e^{2xy} \cdot 2x = \lambda \cdot 3y^2 \\
(3) & \quad x^3 + y^3 = 16
\end{align*}
\]

To solve the first equation for \(\lambda\), we need to divide by \(x\). So we need to consider the cases \(x = 0\) and \(x \neq 0\) separately. If \(x = 0\), the first equation implies \(y = 0\), but putting \(x = 0\) and \(y = 0\) into the third equation gives a contradiction. Therefore \(x = 0\) is impossible. A similar argument shows \(y = 0\) is impossible. So we have \(x \neq 0\) and \(y \neq 0\). Solving the first and second equations for \(\lambda\) gives

\[
\begin{align*}
\lambda &= e^{2xy} \cdot \frac{2y}{3x^2}, \\
\lambda &= e^{2xy} \cdot \frac{2x}{3y^2}.
\end{align*}
\]
Hence
\[ e^{2xy} \cdot \frac{2y}{3x^2} = e^{2xy} \cdot \frac{2x}{3y^2} \]
\[ \frac{2y}{3x^2} = \frac{2x}{3y^2} \]
\[ y^3 = x^3 \]

Putting this into the third equation gives
\[ 2x^3 = 16 \implies x^3 = 8. \]
So
\[ y^3 = x^3 = 8 \implies x = y = 2. \]
So the only critical point is \((2, 2)\) and \(f(2, 2) = e^8\).

5. If you have to use Lagrange multipliers to find the minimum distance between the curve \(x + y \ln x = e^y\) and the point \((2, 1)\), what is the objective function and the constraint?

The distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) is
\[ \text{dist}((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \]

Objective Function: \(f(x, y) = \sqrt{(x - 2)^2 + (y - 1)^2}\).

Constraint: \(x + y \ln x = e^y\)

7. Find an upper bound for the error in using Simpson’s rule with \(n = 10\) to approximate
\[ \int_2^4 \ln(x^2) \, dx. \]

Error Bound for Simpson’s Rule:
\[ E_S \leq \frac{K(b-a)^5}{180n^4} \quad \text{if} \quad |f^{(4)}(x)| \leq K \quad \text{on} \quad [a, b]. \]

\[ f(x) = \ln(x^2) \]
\[ f'(x) = \frac{1}{x^2} \cdot 2x = \frac{2}{x} \]
\[ f''(x) = -\frac{2}{x^2} \]
\[ f^{(3)}(x) = \frac{4}{x^3} \]
\[ f^{(4)}(x) = -\frac{12}{x^4} \]
\[ |f^{(4)}(x)| = \frac{12}{x^4} \]

Since \( |f^{(4)}(x)| = \frac{12}{x^4} \) is decreasing on \([2, 4]\),
\[ |f^{(4)}(x)| = \frac{12}{x^4} \leq \frac{12}{2^4} = \frac{3}{4} \quad \text{on} \quad [2, 4]. \]
So we take $K = \frac{3}{4}$. Therefore

$$E_S \leq \frac{4}{7}(4 - 2)^5 \frac{5}{180(10)^4}.$$ 

8. Suppose a random variable $X$ has probability density function $f(x) = k \cos^2(\pi x)$, $2 \leq x \leq 6$. What is the value of $k$? What is the cumulative distribution function of $X$? What is the expectation of $X$?

$$1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_{2}^{6} k \cos^2(\pi x) \, dx$$

$$k \int_{2}^{6} \frac{1 + \cos(2\pi x)}{2} \, dx = k \left[ \frac{x + \sin(2\pi x)}{2\pi} \right]_{2}^{6}$$

$$= \frac{k}{2} \left( \frac{6 + \sin(12\pi)}{2\pi} - \left( \frac{2 + \sin(4\pi)}{2\pi} \right) \right)$$

$$= \frac{k}{2} (6 + 0) - (2 + 0) = 2k$$

So $k = \frac{1}{2}$.

The cumulative distribution function of $X$ is $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t) \, dt$.

If $x < 2$,

$$F(x) = \int_{-\infty}^{x} f(t) \, dt = \int_{-\infty}^{2} 0 \, dt = 0.$$ 

If $2 \leq x \leq 6$,

$$F(x) = \int_{-\infty}^{x} f(t) \, dt = \int_{-\infty}^{2} f(t) \, dt + \int_{2}^{6} f(t) \, dt$$

$$= \int_{-\infty}^{2} 0 \, dt + \int_{2}^{6} k \cos^2(\pi t) \, dt = \frac{1}{2} \int_{2}^{6} \cos^2(\pi t) \, dt$$

$$= \frac{1}{4} \left[ \frac{x + \sin(2\pi x)}{2\pi} \right]_{2}^{x}$$

$$= \frac{1}{4} \left( \frac{x + \sin(2\pi x)}{2\pi} - \left( \frac{2 + \sin(4\pi)}{2\pi} \right) \right)$$

$$= \frac{1}{4} \left( \frac{x + \sin(2\pi x)}{2\pi} - (2 + 0) \right)$$

$$= x - 2 + \frac{2\sin(2\pi x)}{\pi}.$$ 

If $x > 6$, then

$$F(x) = \int_{-\infty}^{x} f(t) \, dt = \int_{-\infty}^{2} f(t) \, dt + \int_{2}^{6} f(t) \, dt + \int_{6}^{x} f(t) \, dt$$

$$= \int_{-\infty}^{2} 0 \, dt + \int_{2}^{6} k \cos^2(\pi t) \, dt + \int_{6}^{x} 0 \, dt$$

$$= \int_{2}^{6} k \cos^2(\pi t) \, dt = 1.$$
The expectation of $X$ is

$$E(X) = \int_{-\infty}^{\infty} xf(x) \, dx = \frac{1}{2} \int_{2}^{6} x \cos^2(\pi x) \, dx.$$ 

We need to integrate by parts. Let

$$u = x \quad dv = \cos^2(\pi x) \, dx$$

$$du = dx \quad v = \int \cos^2(\pi x) \, dx$$

We compute

$$v = \int \cos^2(\pi x) \, dx = \frac{1}{2} \int (1 + \cos(2\pi x)) \, dx = \frac{1}{2} \left( x + \frac{\sin(2\pi x)}{2\pi} \right).$$

Therefore

$$\int x \cos^2(\pi x) \, dx = uv - \int v \, du$$

$$= \frac{x}{2} \left( x + \frac{\sin(2\pi x)}{2\pi} \right) - \frac{1}{2} \int \left( x + \frac{\sin(2\pi x)}{2\pi} \right) \, dx$$

$$= \frac{x}{2} \left( x + \frac{\sin(2\pi x)}{2\pi} \right) - \frac{x^2}{4} + \frac{\cos(\pi x)}{8\pi^2} + C.$$ 

Hence

$$E(X) = \frac{1}{2} \int_{2}^{6} x \cos^2(\pi x) \, dx$$

$$= \frac{1}{2} \left[ \frac{x}{2} \left( x + \frac{\sin(2\pi x)}{2\pi} \right) - \frac{x^2}{4} + \frac{\cos(2\pi x)}{8\pi^2} \right]_{2}^{6}$$

$$= \frac{1}{2} \left( \frac{1}{2} \left( 6 + \frac{\sin(12\pi)}{2\pi} \right) - \frac{6^2}{4} + \frac{\cos(12\pi)}{8\pi^2} \right) - \frac{1}{2} \left( \frac{1}{2} \left( 2 + \frac{\sin(4\pi)}{2\pi} \right) - \frac{2^2}{4} + \frac{\cos(4\pi)}{8\pi^2} \right)$$

$$= \frac{1}{2} \left( 3 \left( 6 + \frac{1}{8\pi^2} \right) - \frac{6^2}{4} + \frac{1}{8\pi^2} \right) - \frac{1}{2} \left( 2 + \frac{1}{8\pi^2} \right) - \frac{1}{2} \left( 2 + \frac{1}{8\pi^2} \right).$$

9. Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} \ln \left( \frac{k}{2k+5} \right)$$

We have

$$\lim_{k \to \infty} \ln \left( \frac{k}{2k+5} \right) = \lim_{k \to \infty} \ln \left( \frac{k}{2 + \frac{5}{k}} \right) = \ln \left( \frac{1}{2} \right) \neq 0.$$ 

So the series diverges (by the Divergence Test).

10. Determine whether the following series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$$

The Divergence, Comparison, and Ratio Tests don’t work (at least, not very easily). Since $f(x) = \frac{1}{x \sqrt{\ln x}}$ is nonnegative and nonincreasing for $x \geq 2$, we can try the integral test. The function is
easy to integrate:

\[
\int_2^\infty \frac{1}{x \sqrt{\ln x}} \, dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \sqrt{\ln x}} \, dx \\
= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u} \, du \\
= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} u^{-1/2} \, du \\
= \lim_{t \to \infty} \frac{u^{1/2} \ln t}{1/2 \ln 2} \\
= \lim_{t \to \infty} 2(\ln t)^{1/2} - 2(\ln 2)^{1/2} \\
= \infty.
\]

Therefore the series diverges.

11. Determine whether the following series converges or diverges.

\[
\sum_{k=1}^{\infty} \frac{k^{1/3} + 2k^{2/3}}{5k^{1/2} + k^{3/4}}
\]

For large \(k\),

\[
\frac{k^{1/3} + 2k^{2/3}}{5k^{1/2} + k^{3/4}} \approx \frac{2k^{2/3}}{k^{3/4}} \approx k^{2/3} \cdot k^{-3} = k^{-1/2} = \frac{1}{\sqrt{k}}.
\]

Since \(\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}\) diverges (\(p\)-series with \(p \leq 1\)), we expect the original series to diverge. To prove divergence with the Limit Comparison Test, we set \(a_k = \frac{1}{\sqrt{k}}\) and \(b_k = \frac{k^{1/3} + 2k^{2/3}}{5k^{1/2} + k^{3/4}}\) and check that \(\lim_{k \to \infty} \frac{a_k}{b_k}\) converges:

\[
\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{1}{k^{1/2}} \cdot \frac{5k^{1/2} + k^{3/4}}{k^{1/3} + 2k^{2/3}} \\
= \lim_{k \to \infty} \frac{1}{k^{1/2}} \cdot \frac{k^{3/4} \cdot 5k^{-1/4} + 1}{k^{2/3} \cdot k^{-1/3} + 2} \\
= \lim_{k \to \infty} \frac{k^{3/4}}{k^{3/4}} \cdot \frac{5k^{-1/4} + 1}{k^{-1/3} + 2} \\
= \frac{1}{2}.
\]

So the series diverges.

12. Find the MacLaurin series of

\[
f(x) = \frac{1}{(4-x)^2}
\]
We start with the geometric series formula and differentiate both sides:

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1
\]

\[
\frac{1}{(1 - x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \quad \text{for } |x| < 1
\]

Then

\[
f(x) = \frac{1}{(4 - x)^2} = \frac{1}{4^2 (1 - \frac{x}{4})^2}
\]

\[
= \sum_{n=1}^{\infty} n \left(\frac{x}{4}\right)^{n-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{n x^{n-1}}{4^{n+1}} \quad \text{for } \left|\frac{x}{4}\right| < 1, \text{ i.e., } |x| < 4.
\]

As a series starting at \( n = 0 \),

\[
f(x) = \sum_{n=0}^{\infty} \frac{(n + 1)x^n}{4^{n+2}} \quad \text{for } |x| < 4.
\]

13. Use the Macularin series representation for \( \ln(1 - x) \),

\[
\ln(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n},
\]

to find the exact value of the infinite series

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^{n+1}n}.
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^n + 1}{3^n(3n + 1)n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \frac{1}{3} \ln \left( 1 - \left( -\frac{1}{3} \right) \right) = \frac{1}{3} \ln \left( \frac{4}{3} \right).
\]