Representing Functions as Sums of Power Series

We already have power series representations for some important functions.

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x
\]

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x
\]

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x
\]

Geometric Series.

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1
\]

We can use these formulas to find power series representations for related functions.

**Example.** Find a power series representation at 0 for \( x \cos(3\sqrt{x}) \).

Using the power series representation for \( \cos x \) at 0, we have

\[
x \cos(3\sqrt{x}) = x \sum_{n=0}^{\infty} \frac{(-1)^n (3\sqrt{x})^{2n}}{(2n)!} = x \sum_{n=0}^{\infty} \frac{(-1)^n 9^n x^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 9^n x^{n+1}}{(2n)!}
\]

for all \( x \).

**Example.** Find a power series representation at 0 for \( \frac{x^3}{1 + 4x^2} \).

Using the geometric series formula, we have

\[
x^3 = \frac{x^3}{1 - (-4x^2)} = x^3 \sum_{n=0}^{\infty} (-4x^2)^n = \sum_{n=0}^{\infty} (-4)^n x^{2n+3}
\]

for \( | -4x^2 | < 1 \), i.e., \( |x|^2 < \frac{1}{4} \), i.e., \( |x| < \frac{1}{2} \).

**Example.** Find a power series representation at 0 for \( \frac{1}{3x - 5} \).

Using the geometric series formula, we have

\[
\frac{1}{3x - 5} = -\frac{1}{5 - 3x} = -\frac{1}{5} \cdot \frac{1}{1 - \frac{3}{5}x} = -\frac{1}{5} \sum_{n=0}^{\infty} \left( \frac{3}{5}x \right)^n = \sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} x^n \quad \text{for } \left| \frac{3}{5}x \right| < 1 \), i.e., \( |x| < \frac{5}{3} \).
**Example.** Find a power series representation at 0 for

\[ f(x) = \frac{2}{1 - x} + \frac{1}{3x - 5}. \]

We have

\[ \frac{2}{1 - x} = \sum_{n=0}^{\infty} 2x^n \quad \text{for} \ |x| < 1, \]

and

\[ \frac{1}{3x - 5} = \sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} x^n \quad \text{for} \ |x| < \frac{5}{3}. \]

Therefore

\[ f(x) = \frac{2}{1 - x} + \frac{1}{3x - 5} = \sum_{n=0}^{\infty} 2x^n + \sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} x^n = \sum_{n=0}^{\infty} \left( 2 - \frac{3^n}{5^{n+1}} \right) x^n \quad \text{for} \ |x| < 1. \]

**Theorem.** If \( \sum_{n=0}^{\infty} c_n (x - a)^n \) has radius of convergence \( R > 0 \), then

\[
\begin{align*}
(1) \quad & \frac{d}{dx} \left( \sum_{n=0}^{\infty} c_n (x - a)^n \right) = \sum_{n=0}^{\infty} \frac{d}{dx} c_n (x - a)^n = \sum_{n=1}^{\infty} n c_n (x - a)^{n-1}, \\
(2) \quad & \int \left( \sum_{n=0}^{\infty} c_n (x - a)^n \right) \, dx = \sum_{n=0}^{\infty} \int c_n (x - a)^n \, dx = \sum_{n=0}^{\infty} \frac{c_n (x - a)^{n+1}}{n+1} + C,
\end{align*}
\]

and the series in (1) and (2) have radius of convergence \( R \).

**Summary.** Differentiation and integration of power series is done term-by-term and the radius of convergence doesn’t change.

**Remark.** To see why the series on the right-hand side of (1) starts at \( n = 1 \), expand things out:

\[
\frac{d}{dx} \left( \sum_{n=0}^{\infty} c_n (x - a)^n \right) = \sum_{n=0}^{\infty} \frac{d}{dx} c_n (x - a)^n \\
= \frac{d}{dx} c_0 + \frac{d}{dx} c_1 (x - a) + \frac{d}{dx} c_2 (x - a)^2 + \frac{d}{dx} c_3 (x - a)^3 + \cdots \\
= 0 + c_1 + 2c_2 (x - a) + 3c_3 (x - a)^2 + \cdots \\
= \sum_{n=1}^{\infty} n c_n (x - a)^{n-1}.
\]
Example. Differentiating the geometric series formula gives

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1
\]

\[
\frac{d}{dx} \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{d}{dx} x^n \quad \text{for } |x| < 1
\]

\[
\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{for } |x| < 1
\]

\[
\frac{d}{dx} \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} \frac{d}{dx} nx^{n-1} \quad \text{for } |x| < 1
\]

\[
\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \quad \text{for } |x| < 1
\]

\[
\frac{d}{dx} \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{d}{dx} n(n-1)x^{n-2} \quad \text{for } |x| < 1
\]

\[
\frac{6}{(1-x)^4} = \sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3} \quad \text{for } |x| < 1
\]

\[
\vdots
\]

Extra Example. Integrating the geometric series formula gives

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1
\]

\[
\int \frac{1}{1-x} \, dx = \sum_{n=0}^{\infty} \int x^n \, dx \quad \text{for } |x| < 1
\]

\[
-\ln |1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \quad \text{for } |x| < 1
\]

Plugging in \( x = 0 \) gives

\[
\ln |1-0| = \sum_{n=0}^{\infty} \frac{(0)^{n+1}}{n+1} + C,
\]

hence \( C = \ln(1) = 0 \). Moreover, \( 1 - x > 0 \) when \( |x| < 1 \). Therefore

\[
-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{for } |x| < 1.
\]

The formula also holds when \( x = -1 \) (though we didn’t study the tools needed to prove this).

\[
-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{for } -1 \leq x < 1.
\]
Example. Find a power series representation for \( \arctan x \) at 0. We know
\[
\int \frac{1}{1 + x^2} \, dx = \arctan x + C.
\]
So we just need to find a power series representation for \( \frac{1}{1 + x^2} \) and integrate. We have
\[
\frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for } | - x^2 | < 1, \text{ that is, for } |x| < \sqrt{1} = 1
\]
Therefore
\[
\int \frac{1}{1 + x^2} \, dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} \, dx \quad \text{for } |x| < 1
\]
\[
\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1} + C \quad \text{for } |x| < 1
\]
Plugging in \( x = 0 \) gives \( C = \arctan(0) = 0 \). Therefore
\[
\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1} \quad \text{for } |x| < 1
\]
The formula also holds when \( x = \pm 1 \) (though we didn’t study the tools needed to prove this).
\[
\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1} \quad \text{for } |x| \leq 1.
\]

Example. Evaluate
\[
1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}
\]
We recognize this as the series for \( \arctan x \) evaluated at \( x = 1 \):
\[
\arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}
\]
Since \( \arctan(1) = \pi/4 \) (because \( \tan (\pi/4) = 1 \)), we have
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} = \frac{\pi}{4}.
\]

Remark. This gives an approximation for \( \pi \):
\[
\frac{\pi}{4} \approx \sum_{n=0}^{N} \frac{(-1)^n}{2n + 1}.
\]
We can use Taylor’s inequality to bound the error made in this approximation.

Extra Example. Let \( f(x) = \frac{e^x - 1}{x} \) for \( x \neq 0 \) and \( f(0) = 1 \). Show that
\[
\sum_{k=1}^{\infty} \frac{k}{(k + 1)!} = f'(1) = 1.
\]
Using the power series representation of $e^x$ at 0, we have
\[ f(x) = \frac{e^x - 1}{x} = \frac{1}{x} \left[ \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) - 1 \right] = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \]
for all $x \neq 0$. Then
\[ f'(x) = \sum_{n=1}^{\infty} \frac{d}{dx} \frac{x^{n-1}}{n!} = \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{n!} \quad \text{for all } x \neq 0. \]
So
\[ f'(1) = \sum_{n=2}^{\infty} \frac{(n-1)}{n!} = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}. \]
On the other hand, for $x \neq 0$,
\[ f'(x) = \frac{d}{dx} \frac{e^x - 1}{x} = \frac{xe^x - (e^x - 1)}{x^2}, \]
and so
\[ f'(1) = 1. \]

**Theorem.** If $f$ is represented by a power series at $a$, that is, if
\[ f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{for } |x-a| < R \]
with $R > 0$, then the coefficients are
\[ c_n = \frac{f^{(n)}(a)}{n!}. \]
and $\sum_{n=0}^{\infty} c_n(x-a)^n$ is actually the Taylor series of $f$ at $a$.

**Extra Example.** Let $f(x) = \frac{(e^x - 1)}{x}$ for $x \neq 0$ and $f(0) = 1$. Find the Maclaurin series (Taylor series at 0) for $f$. Then find the Maclaurin series for $f'$.

We could try to compute $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, \ldots and look for a pattern to find the Maclaurin series for $f$ using the definition. But it is much easier to use the theorem.

Using the power series representation of $e^x$ at 0, we have
\[ f(x) = \frac{e^x - 1}{x} = \frac{1}{x} \left[ \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) - 1 \right] = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \]
for all $x \neq 0$. When $x = 0$, the series on the the right-hand side equals 1. Since $f(0) = 1$, we actually have
\[ f(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \quad \text{for all } x. \]
By the theorem, this series is the Maclaurin series for $f$.
\[ f'(x) = \sum_{n=1}^{\infty} \frac{d}{dx} \frac{x^{n-1}}{n!} = \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{(n+2)!} \quad \text{for all } x. \]
By the theorem, this series is the Maclaurin series for $f'$. 

Optional: Power Series Representations at $a \neq 0$

So far we have only found power series representations for functions at 0. Similar techniques can be used to find power series representations at other points $a$.

**Example.** Find a power series representation for $\cos x$ at 4.

We will use the trigonometric identity

$$\cos(a + b) = \cos a \cos b - \sin a \sin b.$$  

We will also use the power series representations for $\cos x$ and $\sin x$ at 0, which are valid for all $x$. We have

$$\cos x = \cos(x - 4 + 4) = \cos(x - 4) \cos 4 - \sin(x - 4) \sin 4$$

$$= \cos 4 \sum_{n=0}^{\infty} \frac{(-1)^n(x - 4)^{2n}}{(2n)!} - \sin 4 \sum_{n=0}^{\infty} \frac{(-1)^n(x - 4)^{2n+1}}{(2n + 1)!}$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{k!} (x - 4)^k \quad \text{for all } x,$$

where

$$a_k = \begin{cases} 
\cos 4 & \text{if } k = 4\ell \\
-\sin 4 & \text{if } k = 4\ell + 1 \\
-\cos 4 & \text{if } k = 4\ell + 2 \\
\sin 4 & \text{if } k = 4\ell + 3 
\end{cases}$$

**Example.** Find a power series representation for $\frac{1}{x}$ at $a \neq 0$.

We have

$$\frac{1}{x} = \frac{1}{a + x - a} = \frac{1}{a} \cdot \frac{1}{1 + \frac{x - a}{a}} = \frac{1}{a} \cdot \frac{1}{1 - \left(\frac{x - a}{a}\right)}$$

$$= \frac{1}{a} \sum_{n=0}^{\infty} \left(-\frac{x - a}{a}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (x - a)^n$$

for $\left|\frac{x - a}{a}\right| < 1$, i.e., $|x - a| < |a|$.