**Numerical Integration**

It is sometimes difficult or impossible to find the exact value of a definite integral. For example, it is impossible to evaluate the following integrals exactly:

\[
\int_{0}^{1} e^{x^2} \, dx \quad \int_{-1}^{1} \sqrt{1 + x^3} \, dx
\]

As another example, if a function is determined from collected data, there may be no formula for the function and hence no way to compute a definite integral of the function exactly. When we can’t compute a definite integral exactly, we need to approximate it.

The definite integral of \(f\) on \([a,b]\) is a limit of Riemann sums of \(f\) on \([a,b]\):

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(\bar{x}_i) \Delta x,
\]

where \(\Delta x = \frac{b-a}{n}, \, x_i = a + i \Delta x\), and \(\bar{x}_i\) is in \([x_{i-1}, x_i]\).

Any Riemann sum for \(f\) on \([a,b]\) is an approximation to the definite integral of \(f\) on \([a,b]\):

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{i=1}^{n} f(\bar{x}_i) \Delta x
\]

For a midpoint Riemann sum, \(\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i] \text{ for } i = 1, \ldots, n\). The approximation obtained using a midpoint Riemann sum is called the Midpoint Rule approximation to \(\int_{a}^{b} f(x) \, dx\).

### Midpoint Rule

\[
\int_{a}^{b} f(x) \, dx \approx M(n) = \sum_{i=1}^{n} f \left( \frac{x_{i-1} + x_i}{2} \right) \Delta x
\]

\[
= \Delta x \left[ f \left( \frac{x_0 + x_1}{2} \right) + f \left( \frac{x_1 + x_2}{2} \right) + \cdots + f \left( \frac{x_{n-1} + x_n}{2} \right) \right]
\]

where \(\Delta x = \frac{b-a}{n}\) and \(x_i = a + i \Delta x\) for \(i = 0, \ldots, n\).
A Riemann sum approximation to \( \int_{a}^{b} f(x)\,dx \) works by approximating the region between \( y = f(x) \), the \( x \)-axis, \( x = a \), and \( x = b \) by rectangles.

We obtain a different approximation to \( \int_{a}^{b} f(x)\,dx \) if we use trapezoids instead of rectangles:

The area of the \( i \)th trapezoid is \( \frac{1}{2} (f(x_{i-1}) + f(x_{i})) \Delta x \), so the trapezoid approximation to \( \int_{a}^{b} f(x)\,dx \) is

\[
\int_{a}^{b} f(x)\,dx \approx T(n) = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_{i})}{2} \Delta x
\]

where \( \Delta x = \frac{b-a}{n} \) and \( x_{i} = a + i \Delta x \) for \( i = 0, \ldots, n \).
When approximating the region between \( y = f(x) \), the \( x \)-axis, \( x = a \), and \( x = b \) by trapezoids, we used the line segment between \( f(x_{i-1}) \) and \( f(x_i) \) to form the \( i \)th trapezoid. In other words, we approximated the graph of \( f \) by line segments.

We obtain another approximation if we use parabolic arcs instead of line segments to approximate the graph of \( f \):

\[
\begin{align*}
\text{Simpson's Rule} \\
\int_a^bf(x)dx & \approx S(n) = \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]
\end{align*}
\]

where \( n \) is even, \( \Delta x = \frac{b-a}{n} \), and \( x_i = a + i\Delta x \) for \( i = 0, \ldots, n \).

Remark

- In a Riemann sum approximation, we are approximating \( f \) by a constant function (degree 0 polynomial) on each interval \([x_{i-1}, x_i]\).
- In the trapezoidal approximation, we are approximating \( f \) by a linear function (degree 1 polynomial) on each interval \([x_{i-1}, x_i]\).
- In Simpson’s approximation, we are approximating \( f \) by a quadratic function (degree 2 polynomial) on each interval \([x_{i-1}, x_i]\).
\textbf{Example} Use the trapezoid rule with \( n = 3 \) subintervals to approximate \( \int_{0}^{3} \frac{1}{1 + x^3} \, dx \).

\[ \Delta x = \frac{b - a}{n} = \frac{3 - 0}{3} = 1 \]

\[ x_i = a + i \Delta x = 0 + i \cdot 1 = i \]

\[ x_0 = 0 \]
\[ x_1 = 1 \]
\[ x_2 = 2 \]
\[ x_3 = 3 \]

\[ f(x) = \frac{1}{1 + x^3} \]

Therefore

\[ \int_{0}^{3} \frac{1}{1 + x^3} \, dx \approx \Delta x \left[ \frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \frac{1}{2} f(x_3) \right] \]

\[ = 1 \cdot \left[ \frac{1}{2} \cdot \frac{1}{1 + 0^3} + \frac{1}{1 + 1^3} + \frac{1}{1 + 2^3} + \frac{1}{2} \cdot \frac{1}{1 + 3^3} \right] \quad \text{(you can stop here on the exams)} \]

\[ = \frac{1}{2} + \frac{1}{2} + \frac{1}{9} + \frac{1}{56} = \frac{569}{504} \]

\[ = 1.1289 \ldots \]
An approximation is not very useful without some information about its accuracy.

**Example 3.** 3.14, 3.14159, and 10 are all approximations to $\pi$. Some of these approximations are more accurate than others.

If $c$ is an approximate numerical solution to a problem having an exact solution $x$, then

$$\text{absolute error} = |x - c|$$

$$\text{relative error} = \frac{|x - c|}{|x|} \quad (\text{if } x \neq 0)$$

When the value of the exact solution $x$ is not be known, an upper bound on the absolute or relative error is very valuable.

**Example** If we know that $c = 1.3191$ is an approximate solution to the equation $e^x = x^2 + 2$ and the absolute error of this approximation satisfies $|x - c| \leq 0.0002$, then the exact solution $x$ satisfies

$$c - 0.0002 \leq x \leq c + 0.0002$$

$$1.3189 \leq x \leq 1.3193$$

**Midpoint Rule Error Bound**

If we can find a number $K$ such that $|f''(x)| \leq K$ for all $x$ in $[a, b]$, then the absolute error in approximating $\int_a^b f(x)dx$ using the Midpoint Rule with $n$ subintervals satisfies

$$E_M(n) = \left| \int_a^b f(x)dx - M(n) \right| \leq \frac{K(b - a)^3}{24n^2}$$

**Trapezoid Rule Error Bound**

If we can find a number $K$ such that $|f''(x)| \leq K$ for all $x$ in $[a, b]$, then the absolute error in approximating $\int_a^b f(x)dx$ using the Trapezoid Rule with $n$ subintervals satisfies

$$E_T(n) = \left| \int_a^b f(x)dx - T(n) \right| \leq \frac{K(b - a)^3}{12n^2}$$

**Simpson’s Rule Error Bound**

If we can find a number $K$ such that $|f^{(4)}(x)| \leq K$ for all $x$ in $[a, b]$, then the absolute error in approximating $\int_a^b f(x)dx$ using Simpson’s rule with $n$ subintervals satisfies

$$E_S(n) = \left| \int_a^b f(x)dx - S(n) \right| \leq \frac{K(b - a)^5}{180n^4}$$
Example How large should we take $n$ in order to guarantee that the Simpson’s rule approximation to 
\[
\int_1^2 \frac{1}{x^3} \, dx
\]
is accurate to $0.000000000002 = 2 \cdot 10^{-12}$?

A 10  
B 100  
C 1000  
D 10000  
E 100000

We want 
\[
E_{S(n)} = \left| \int_1^2 \frac{1}{x^3} \, dx - S(n) \right| \leq 2 \cdot 10^{-12}.
\]

Note that 
\[
\begin{align*}
f(x) &= x^{-3}  
\quad f'(x) = -3x^{-4}  
\quad f''(x) = 12x^{-5}  
\quad f^{(3)}(x) = -60x^{-6}  
\quad f^{(4)}(x) = 360x^{-7}
\end{align*}
\]

Since $0 < \frac{1}{x} \leq 1$ on $[1, 2]$, we have 
\[
\left| f^{(4)}(x) \right| = \frac{360}{x^7} \leq 360 \quad \text{on} \quad [1, 2].
\]

Therefore 
\[
E_{S(n)} \leq \frac{K(b-a)^5}{180n^4} = \frac{360(2-1)^5}{180n^4} = \frac{2}{n^4}.
\]

So it suffices to have 
\[
\frac{2}{n^4} \leq 2 \cdot 10^{-12}  
n^4 \geq 10^{12}  
n \geq 10^3 = 1000
\]

Remark 
\[
\int_1^2 \frac{1}{1 + x^3} \, dx = \frac{3}{8} = 0.375
\]
\[
\begin{align*}
S(10) &= 0.3750312659228 \ldots & (\text{Time (seconds): 0.000} \ldots) \\
S(100) &= 0.3750000032795 \ldots & (\text{Time (seconds): 0.031} \ldots)  \\
S(1000) &= 0.3750000000003 \ldots & (\text{Time (seconds): 0.049} \ldots)  \\
S(10000) &= 0.3750000000000003 \ldots & (\text{Time (seconds): 1.435} \ldots)  \\
S(100000) &= 0.375000000000000003 \ldots & (\text{Time (seconds): 135.674} \ldots)
\end{align*}
\]