An antiderivative of \( f \) on an interval \( I \) is a function \( F \) such that
\[
F'(x) = f(x) \quad \text{for all } x \text{ in } I.
\]

If \( F \) is an antiderivative of \( f \) on \( I \), then the family of all antiderivatives of \( f \) on \( I \) consists of the functions of the form
\[
G(x) = F(x) + C
\]
where \( C \) is any constant.

\[
\frac{d}{dx} G(x) = \frac{d}{dx} (F(x) + C) = \frac{d}{dx} F(x) = f(x)
\]

The indefinite integral of \( f \) (on \( I \)), denoted by \( \int f(x) \, dx \), is the family of all antiderivatives of \( f \) (on \( I \)):
\[
\int f(x) \, dx = F(x) + C,
\]
where \( F \) is an antiderivative of \( f \) (on \( I \)) and \( C \) is an arbitrary constant.
• \( \frac{d}{dx} \int f(x) \, dx = f(x) \)

• \( \int \frac{d}{dx} f(x) \, dx = f(x) + C \)

• \( \int f(x) \, dx = F(x) + C \iff \frac{d}{dx} F(x) = f(x) \)

Table of Indefinite Integrals

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)
\]

because \( \frac{d}{dx} \left( \frac{x^{n+1}}{n+1} \right) = \frac{(n+1)x^n}{n+1} = x^n \) for \( n \neq -1 \).

\[
\int \frac{1}{x} \, dx = \ln|x| + C
\]

because \( \frac{d}{dx} (\ln|x|) = \frac{1}{x} \).

\[
\int e^x \, dx = e^x + C
\]

because \( \frac{d}{dx} (e^x) = e^x \).
\[ \int \ln x \, dx = x \ln x - x + C \]

because \[ \frac{d}{dx}(x \ln x - x) = \left( \frac{dx}{dx} \right) \ln x + x \cdot \left( \frac{d}{dx} \ln x \right) - x \]

\[ = \ln x + x \cdot \frac{1}{x} - 1 \]

\[ = \ln x + 1 - 1 \]

\[ = \ln x \]

(or use integration by parts, which we'll learn about soon)

\[ \int \sin x \, dx = -\cos x + C \]

because \[ \frac{d}{dx}(-\cos x) = \sin x \]

\[ \int \cos x \, dx = \sin x + C \]

because \[ \frac{d}{dx}(\sin x) = \cos x \]

\[ \int \sec^2 x \, dx = \tan x + C \]

because \[ \frac{d}{dx} \tan x = \sec^2 x \]

\[ \int \sec x \tan x \, dx = \sec x + C \]

because \[ \frac{d}{dx} \sec x = \sec x \tan x \]

\[ \int \csc^2 x \, dx = \cot x + C \]

because \[ \frac{d}{dx}(-\cot x) = \csc^2 x \]

\[ \int \csc x \cot x \, dx = -\csc x + C \]

because \[ \frac{d}{dx}(-\csc x) = \csc x \tan x \]
\[ \int \tan x \, dx = \ln |\sec x| + C \]

(Check that \( \frac{d}{dx} \ln |\sec x| = \tan x \) or use the substitution rule, which we'll learn about soon)

\[ \int \sec x \, dx = \ln |\sec x + \tan x| + C \]

(Check that \( \frac{d}{dx} \ln |\sec x| = \tan x \) or use the substitution rule, which we'll learn about soon)

\[ \int \frac{1}{1 + x^2} \, dx = \arctan(x) + C \]

because \( \frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2} \)

By substitution: \( \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C \)

\[ \int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin(x) + C \]

because \( \frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - x^2}} \)

By substitution: \( \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \left( \frac{x}{a} \right) + C \)
The definite integral \( \int_a^b f(x) \, dx \) is a number (the net area between \( y = f(x) \), the x-axis, \( x=a \), and \( x=b \)).

The indefinite integral \( \int f(x) \, dx \) is a family of functions (the family of all antiderivatives of \( f \))

\[
\int f(x) \, dx = F(x) + C
\]

where \( F \) is an antiderivative of \( f \) and \( C \) is an arbitrary constant.

The area function of \( f \) with left endpoint \( c \) is defined to be

\[
A_c(x) = \int_c^x f(t) \, dt
\]

= net area between the graph of \( f \), and the horizontal axis from \( c \) to the variable point \( x \).
Ex. Let $f(x) = 2x - 3$. Compute the area function of $f$ with left endpoint $c = -1$,

$$A_{-1}(x) = \int_{-1}^{x} (2t - 3) \, dt,$$

(a) using a limit of Riemann sums
(b) using the geometric interpretation of the integral

(a) $A_{-1}(x) = \lim_{n \to \infty} \sum_{i=1}^{n} (2t_i - 3) \Delta t$

$$\Delta t = \frac{x - (-1)}{n} = \frac{x + 1}{n}$$

$$t_i = a + i \Delta t = -1 + \frac{i(x + 1)}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left( 2(i \Delta t) - 3 \right) \left( \frac{x + 1}{n} \right)$$

$$= \lim_{n \to \infty} \left( \frac{x + 1}{n} \right) \sum_{i=1}^{n} \left( -2 + \frac{2(x + 1)}{n} \right) i - 3$$

$$= \lim_{n \to \infty} \left( \frac{x + 1}{n} \right) \left( \frac{\sum_{i=1}^{n} (-5) + 2(x + 1) \sum_{i=1}^{n} i}{n} \right)$$

$$= \lim_{n \to \infty} \left( \frac{x + 1}{n} \right) \left( -5n + \frac{2(x + 1)}{n} \cdot \frac{n(n+1)}{2} \right)$$

$$= \lim_{n \to \infty} \left( \frac{x + 1}{n} \right) \left( -5 + \frac{(x + 1)(n^2 + n)}{n^2} \right)$$

$$= \lim_{n \to \infty} \left( \frac{x + 1}{n} \right) \left( -5 + \frac{(x + 1)}{1 + \frac{1}{n}} \right)$$

$$= (x + 1) \left( -5 + (x + 1) \left( 1 + \frac{1}{n} \right) \right)$$

$$= (x + 1)^2 - 5(x + 1)$$

$$= x^2 + 2x + 1 - 5x - 5 = x^2 - 3x - 4$$
(b) \( f(\cdot) = 2\cdot + 3 \)

\[
A(c(x)) = \int_{-1}^{x} (2\cdot + 3) \, dx = A_1 - A_2 = \frac{1}{2}(base)(height)
\]

\[
= \frac{1}{2} \left( x - \frac{3}{2} \right) f(x) + \frac{1}{2} \left( \frac{3}{2} - (-1) \right) f(-1)
\]

(We can simply add the signed areas of the individual triangles.)

\[
= \frac{1}{2} \left( x - \frac{3}{2} \right) \left( 2x - 3 \right) + \frac{1}{2} \left( \frac{5}{2} \right) (-5)
\]

\[
= \frac{1}{2} \left( 2x^2 - 3x - 3x + \frac{9}{2} \right) + \frac{1}{2} \left( -\frac{25}{2} \right)
\]

\[
= \frac{1}{2} \left( 2x^2 - 6x + \frac{9}{2} - \frac{25}{2} \right)
\]

\[
= \frac{1}{2} \left( 2x^2 - 6x - 8 \right)
\]

\[
= x^2 - 3x - 4
\]
Fundamental Theorem of Calculus - Part I

If \( f \) is continuous on \([a, b]\), then \( A_a \) (the area function of \( f \) with left endpoint \( a \)) is an antiderivative of \( f \) on \([a, b]\).

In other words,

\[
A'_a(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x) \quad \text{for all } x \in [a, b]
\]

(In fact, for any \( c \) in \([a, b]\), \( A_c \) is an antiderivative of \( f \) on \([a, b]\):

\[
A'_c(x) = \frac{d}{dx} \int_c^x f(t) dt = f(x) \quad \text{for all } x \in [a, b]
\]

**Example** Let \( f(x) = 2x - 3 \). From the previous example, we know \( A_{-1}(x) = x^2 - 3x - 4 \). Note that

\[
A'_{-1}(x) = \frac{d}{dx} (x^2 - 3x - 4) = 2x - 3 = f(x).
\]

This is no coincidence; it's the Fundamental Theorem of Calculus (part I)!