Convergence Tests

For a series \( \sum_{n=1}^{\infty} a_n \), we have

\[
a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = s_n - s_{n-1} \quad \text{for } n > 1.
\]

If \( \sum_{n=1}^{\infty} a_n \) converges, then the sequence of partial sums \( s_n \) converges to a limit \( s \), and so

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1} = s - s = 0.
\]

**Theorem.** If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \).

From this theorem we get:

**Test for Divergence.** If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.

**Example.** Does \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2 + 4}} \) converge or diverge?

We have

\[
\lim_{n \to \infty} \frac{n}{\sqrt{3n^2 + 4}} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2(3 + \frac{4}{n^2})}} = \lim_{n \to \infty} \frac{n}{n\sqrt{3 + \frac{4}{n^2}}} = \lim_{n \to \infty} \frac{1}{\sqrt{3 + \frac{4}{n^2}}} = \frac{1}{\sqrt{3 + 0}} = \frac{1}{\sqrt{3}} \neq 0
\]

So the series diverges by the Test for Divergence.

**Example.** Does \( \sum_{n=1}^{\infty} \cos n \) converge or diverge?

We have

\[
\lim_{n \to \infty} \cos n \ \text{does not exist}
\]

So the series diverges by the Test for Divergence.

**Example.** Does \( \sum_{n=1}^{\infty} \ln \left( \frac{1}{n} \right) \) converge or diverge?

We have

\[
\lim_{n \to \infty} \ln \left( \frac{1}{n} \right) = -\infty
\]

So the series diverges by the Test for Divergence.

**Remark.** The Test for Divergence should be the first test you try. Every divergent series we have seen so far can be shown to be divergent using the Test for Divergence.
Warning. If \( \lim_{n \to \infty} a_n \neq 0 \), we cannot conclude anything about the convergence or divergence of \( \sum_{n=1}^{\infty} a_n \).

For example, the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent even though \( \lim_{n \to \infty} \frac{1}{n} = 0 \). The next test will allow us to prove this.

**Integral Test.** Suppose \( f(x) \) is nonnegative and nonincreasing for \( x \geq 1 \).

- If \( \int_{1}^{\infty} f(x) \, dx \) converges, then \( \sum_{n=1}^{\infty} f(n) \) converges.
- If \( \int_{1}^{\infty} f(x) \, dx \) diverges, then \( \sum_{n=1}^{\infty} f(n) \) diverges.

**\( p \)-Series.**

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.
\]

The case \( p = 1 \) is the harmonic series.

**Proof:** If \( p < 0 \), then \( \lim_{n \to \infty} \frac{1}{n^p} = \infty \). If \( p = 0 \), then \( \lim_{n \to \infty} \frac{1}{n^p} = 1 \). In either case, the \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) diverges by the Test for Divergence. Consider \( p > 0 \). The function \( f(x) = \frac{1}{x^p} \) is clearly nonnegative and nonincreasing for \( x \geq 1 \). While studying improper integrals we found that

\[
\int_{1}^{\infty} \frac{1}{x^p} \, dx \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.
\]

So the Integral Test implies \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges if \( p > 1 \) and diverges if \( 0 \leq p \leq 1 \).

**Example.** Does \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) converge or diverge?

We have

\[
\lim_{n \to \infty} \frac{1}{n \ln n} = 0,
\]

so the Test for Divergence tells us nothing.

The function \( f(x) = \frac{1}{x \ln x} \) is clearly nonnegative and nonincreasing for \( x \geq 2 \). To check that it is nonincreasing for \( x \geq 2 \), observe that

\[
f'(x) = -(x \ln x)^2 (\ln x + 1) = -\frac{\ln x + 1}{(x \ln x)^2} \leq 0 \quad \text{for } x \geq 2.
\]
We have
\[ \int_2^\infty \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \ln x} \, dx \]
\[ = \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u} \, du = \ln x, \, du = \frac{dx}{x} \]
\[ = \lim_{t \to \infty} \ln |u|_{\ln 2}^{\ln t} \]
\[ = \lim_{t \to \infty} (\ln |\ln t| - \ln |\ln 2|) \]
\[ = \infty \]
So the Integral Test implies \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) diverges.

Example. Does \( \sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \) converge or diverge?
\( f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}} \) is nonnegative and nonincreasing for \( x \geq 1 \).
We have
\[ \int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx = \lim_{t \to \infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx \]
\[ = \lim_{t \to \infty} \int_1^{\sqrt{t}} 2e^{-u} \, du = \sqrt{x}, \, du = \frac{1}{2\sqrt{x}} \, dx \]
\[ = -2 \lim_{t \to \infty} \left( e^{-\sqrt{t}} - e^{-1} \right) \]
\[ = 2e^{-1}. \]
So the integral converges. Therefore the Integral Test implies \( \sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \) converges.

Example. Does \( \sum_{n=1}^{\infty} \frac{n}{(n^2 - 7)^3} \) converge or diverge?
Let \( f(x) = \frac{x}{(x^2 - 7)^3} \). Note \( f(x) \) is nonnegative for \( x > \sqrt{7} \). We have
\[ f'(x) = \frac{d}{dx} \left( x(x^2 - 7)^{-3} \right) = (x^2 - 7)^{-3} + x(-3(x^2 - 7)^{-4})(2x) \]
\[ = \frac{x^2 - 7 - 6x^2}{(x^2 - 7)^4} = \frac{-7 - 5x^2}{(x^2 - 7)^4} \leq 0 \quad \text{for } x > \sqrt{7}. \]
So \( f(x) \) is nonincreasing for \( x > \sqrt{7} \). Note \( 3 > \sqrt{7} \).
We compute
\[
\int_{3}^{\infty} \frac{x}{(x^2 - 7)^3} \, dx = \lim_{t \to \infty} \int_{3}^{t} \frac{x}{(x^2 - 7)^3} \, dx
\]
\[
= \lim_{t \to \infty} \frac{1}{2} \int_{2}^{t^2-7} \frac{1}{u^3} \, du = \frac{1}{2} \left[ \frac{1}{u^2} \right]_{t^2-7}^{\infty}
\]
\[
= -\frac{1}{4} \lim_{t \to \infty} \left( \frac{1}{(t^2 - 7)^2} - \frac{1}{2^2} \right)
\]
\[
= \frac{1}{16}.
\]

So the integral converges. Therefore the Integral Test implies \( \sum_{n=3}^{\infty} \frac{n}{(n^2 - 7)^3} \) converges. Since
\[
\sum_{n=1}^{\infty} \frac{n}{(n^2 - 7)^3} = \frac{1}{(2^2 - 7)^3} + 2 \sum_{n=3}^{\infty} \frac{n}{(n^2 - 7)^3},
\]
we conclude \( \sum_{n=1}^{\infty} \frac{n}{(n^2 - 7)^3} \) converges.

The next test allows us to determine whether a series converges by comparing it to another series whose convergence status is known.

**Comparison Test.** Suppose \( 0 \leq a_n \leq b_n \) for all sufficiently large \( n \).

- If \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.
- If \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} b_n \) diverges.

**Example.** Does \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) converge or diverge?

We have
\[
\frac{\ln n}{n} \geq \frac{1}{n} \quad \text{for all } n \geq 3
\]
and \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. Therefore \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) diverges by the Comparison Test.

**Remark.** We could have used the Integral Test in the previous example, but the Comparison Test was easier because of our knowledge of \( p \)-series.

**Example.** Does \( \sum_{n=1}^{\infty} \frac{5n - 1}{2n^3 - 4n + 3} \) converge or diverge?

Note \( \frac{5n - 1}{2n^3 - 4n + 3} \geq 0 \) for \( n \geq 1 \).
For large $n$, the dominant term in the numerator is $5n$, and the dominant term in the denominator is $2n^3$. So

$$\frac{5n - 1}{2n^3 - 4n + 3} \approx \frac{5n}{2n^3} \approx \frac{1}{n^2} \quad \text{for large } n.$$ 

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p$-series with $p > 1$), we expect $\sum_{n=1}^{\infty} \frac{5n - 1}{2n^3 - 4n + 3}$ to converge. We can prove this with the Comparison Test if we show

$$\frac{5n - 1}{2n^3 - 4n + 3} \leq c \frac{1}{n^2} \quad \text{for all sufficiently large } n.$$ 

We work backwards to reduce this inequality to a more obvious one:

$$\frac{5n - 1}{2n^3 - 4n + 3} \leq c \frac{1}{n^2}$$

$$5n^3 - n^2 \leq 2cn^3 - 4cn + 3c$$

$$0 \leq (2c - 5)n^3 + n^2 - 4cn + 3c$$

$$0 \leq (2c - 5)n^3 + (n - 4c)n + 3c$$

The last inequality is true if $c = \frac{5}{2}$ and $n \geq 4c$. Therefore

$$\frac{5n - 1}{2n^3 - 4n + 3} \leq \frac{5}{2} \cdot \frac{1}{n^2} \quad \text{for } n \geq 4 \cdot \frac{5}{2} = 10$$

So, since $\sum_{n=1}^{\infty} \frac{5}{2} \cdot \frac{1}{n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the Comparison Test implies $\sum_{n=1}^{\infty} \frac{5n - 1}{2n^3 - 4n + 3}$ converges.

The next test is often easier to apply than the Comparison Test. It is obtained by combining the Comparison Test with the following observation: If $a_n \geq 0$ and $b_n > 0$ for all sufficiently large $n$, and if $\lim_{n \to \infty} \frac{a_n}{b_n}$ converges, then there is a $c > 0$ such that $a_n \leq cb_n$ for all sufficiently large $n$.

**Limit Comparison Test.** Suppose $a_n \geq 0$ and $b_n > 0$ for all sufficiently large $n$, and suppose $\lim_{n \to \infty} \frac{a_n}{b_n}$ converges.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

**Example.** Does $\sum_{n=1}^{\infty} \frac{5n - 1}{2n^3 - 4n + 3}$ converge or diverge?

For large $n$, the dominant term in the numerator is $5n$, and the dominant term in the denominator is $2n^3$. So

$$\frac{5n - 1}{2n^3 - 4n + 3} \approx \frac{5n}{2n^3} \approx \frac{1}{n^2} \quad \text{for large } n.$$
Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges (\( p \)-series with \( p > 1 \)), we expect \( \sum_{n=1}^{\infty} \frac{5n - 1}{2n^3 - 4n + 3} \) to converge. To prove this with the Limit Comparison Test, we set \( a_n = \frac{5n - 1}{2n^3 - 4n + 3} \) and \( b_n = \frac{1}{n^2} \). Then
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{5n - 1}{2n^3 - 4n + 3}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{5n^3 - n^2}{2n^4 - 4n + 3} = \frac{5}{2}.
\]
So the limit exists. Since \( \sum_{n=1}^{\infty} \frac{5n - 1}{2n^3} \) converges, the Limit Comparison Test implies \( \sum_{n=1}^{\infty} \frac{5n - 1}{2n^3 + 4n + 3} \) converges.

**Example.** Does \( \sum_{n=1}^{\infty} \frac{n}{(n^4 - 9)^{1/3}} \) converge or diverge?

For large \( n \),
\[
\frac{n}{(n^4 - 9)^{1/3}} \approx \frac{n}{n^{4/3}} = \frac{1}{n^{1/3}}.
\]
Since \( \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \) diverges (\( p \)-series with \( p \leq 1 \)), we expect \( \sum_{n=1}^{\infty} \frac{n}{(n^4 - 9)^{1/3}} \) to diverge. To prove this with the Limit Comparison Test, we set \( a_n = \frac{1}{n^{1/3}} \) and \( b_n = \frac{n}{(n^4 - 9)^{1/3}} \). Then
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{n^{1/3}}}{\frac{n}{(n^4 - 9)^{1/3}}} = \lim_{n \to \infty} \frac{(n^4 - 9)^{1/3}}{n^{4/3}} = \lim_{n \to \infty} \frac{n^{4/3} (1 - \frac{9}{n^4})}{n^{4/3}} = 1.
\]
So the limit exists. Since \( \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \) diverges, the Limit Comparison Test implies \( \sum_{n=1}^{\infty} \frac{n}{(n^4 - 9)^{1/3}} \) diverges.

The Integral, Comparison, and Limit Comparison Tests are restricted to series with (eventually) non-negative terms. The next theorem lets us deal with some series that have (infinitely many) negative terms.

**Theorem.** If \( \sum_{n=1}^{\infty} |a_n| \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

If \( \sum_{n=1}^{\infty} |a_n| \) converges, we say \( \sum_{n=1}^{\infty} a_n \) converges absolutely.
Example. Does \( \sum_{n=1}^{\infty} \frac{\sin n}{2^n} \) converge or diverge?

Since \( \left| \frac{\sin n}{2^n} \right| \leq \frac{1}{2^n} \)

and since \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) converges (geometric series with \( |r| < 1 \)), the Comparison Test implies \( \sum_{n=1}^{\infty} \frac{\sin n}{2^n} \) converges. Therefore \( \sum_{n=1}^{\infty} \frac{\sin n}{2^n} \) converges absolutely.

If the \( n \)th term of a series is a product involving factors like \( r^n \) or \( n! \), the following test is often useful.

**Ratio Test.**

- If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \), then \( \sum_{n=1}^{\infty} a_n \) converges absolutely.
- If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \) or \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \), then \( \sum_{n=1}^{\infty} a_n \) diverges.
- Otherwise the Ratio Test is inconclusive.

Example. Does \( \sum_{n=1}^{\infty} \frac{(-5)^n}{n!} \) converge or diverge?

We compute

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-5)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-5)^n} \right| = \lim_{n \to \infty} \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \lim_{n \to \infty} \frac{5^{n+1}}{n!(n+1)} = \lim_{n \to \infty} \frac{5n!}{n!(n+1)} = \lim_{n \to \infty} \frac{5}{n+1} = 0 < 1.
\]

Therefore \( \sum_{n=1}^{\infty} \frac{(-5)^n}{n!} \) converges absolutely by the Ratio Test.
Example. Does $\sum_{n=1}^{\infty} \frac{(2n)!}{n^2}$ converge or diverge?

We compute

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(2(n+1))!}{(n+1)^2} \frac{(n+1^2)}{(2n)!}$$

$$= \lim_{n \to \infty} \frac{(2n+2)!}{(2n)!} \frac{n^2}{(n+1)^2}$$

$$= \lim_{n \to \infty} \frac{(2n+1)(2n+2)}{(2n)!} \frac{n^2}{(n+1)^2}$$

$$= \lim_{n \to \infty} \frac{n^2}{(n+1)^2}$$

$$= \infty$$

Therefore $\sum_{n=1}^{\infty} \frac{(2n)!}{n^2}$ diverges by the Ratio Test.

Example. Does $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3n}{\sqrt{n}}$ converge or diverge?

We compute

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3^{n+1}}{\sqrt{n+1}}$$

$$= \lim_{n \to \infty} \frac{3^n \sqrt{n}}{3^n \sqrt{n+1}}$$

$$= \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= 3 > 1$$

Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3n}{\sqrt{n}}$ diverges by the Ratio Test.

Remark. When the Ratio Test implies divergence, the Test for Divergence implies divergence as well.