Differential equations are used throughout the physical and social sciences. For example, the Black-Scholes equation is a (partial) differential equation that is used extensively in stock options trading; it describes the ideal relationship between the price of a stock option, the price of the stock, and time in the Black-Scholes model of a financial market.

A differential equation is an equation involving an unknown function and some of its derivatives. The order of a differential equation is the order of the highest derivative that occurs in the equation.

\[
\frac{d^3y}{dx^3} + x \frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + x^3y + x^4 = 0
\]

3rd order

\[
y^2 + \left( \frac{dy}{dx} \right)^2 = 6 \cos x
\]

1st order

A solution of a differential equation is a function that satisfies the differential equation (on some interval).

Example. The function \( y = \ln(x^2 + x) \) is a solution of the DE

\[\frac{dy}{dx} = (2x + 1)e^{-y}.\]

Check:

\[\frac{dy}{dx} = \frac{d}{dx} \ln(x^2 + x) = \frac{1}{x^2 + x} \cdot (2x + 1).\]

\[ (2x + 1)e^{-y} = (2x + 1)e^{-\ln(x^2+x)} = \frac{2x + 1}{x^2 + x} \]

So

\[ y' = (2x + 1)e^{-y}. \]

The general solution of a differential equation (on some interval) is the family of all solutions of the differential equation (on that interval). Solving a differential equation means finding the general solution.

A separable differential equation is a first order differential equation of the form

\[ \frac{dy}{dx} = g(x)h(y) \]

We can solve separable differential equations by writing

\[ \frac{dy}{h(y)} = g(x)dy \]

and integrating both sides.
Example. Solve $e^{-x^2+1}y' = -xy^2$

We need to divide by $y$, so we consider two cases.

Case $y(x) = 0$: The constant function $y(x) = 0$ clearly satisfies $e^{-x^2+1}y' = -xy^2$.

Case $y(x) \neq 0$:

$$e^{-x^2+1} \frac{dy}{dx} = -xy^2$$

$$\frac{dy}{dx} = -xe^{-2}y^2$$

$$y^{-2}dy = -xe^{-2}dx$$

$$\int y^{-2}dy = \int -xe^{-2}dx$$

$$-y^{-1} = -\frac{1}{2} \int e^u du \quad (u = x^2 - 1, \ du = 2x \, dx)$$

$$y^{-1} = \frac{1}{2} e^u + C \quad (C \text{ is an arbitrary constant})$$

$$\frac{1}{y} = \frac{1}{2} e^{x^2-1} + C$$

$$y(x) = \frac{1}{\frac{1}{2} e^{x^2-1} + C}$$

where $C$ is an arbitrary constant.

Therefore the general solution consists of the functions

$$y(x) = 0 \quad \text{and} \quad y(x) = \frac{1}{\frac{1}{2} e^{-x^2+1} + C} \quad \text{where } C \text{ is any real number}.$$

Remark. The position of the constant $C$ is important. The function $y(x) = \frac{1}{\frac{1}{2} e^{x^2-1} + C}$ is NOT a solution of $e^{x^2+1}y' = -xy^2$ for $C \neq 0$:

If $y(x) = \frac{1}{\frac{1}{2} e^{x^2-1} + C}$, we have

$$e^{-x^2+1}y' = e^{-x^2+1} \frac{d}{dx} 2e^{-x^2+1} = e^{-x^2+1}(-4xe^{-x^2+1}) = -4xe^{-2x^2+2}$$

and

$$-xy^2 = -x \left( \frac{1}{\frac{1}{2} e^{x^2-1} + C} \right)^2 = -x \left( 2e^{-x^2+1} + C \right)^2 = -4xe^{-2x^2+2} - 4Cxe^{-x^2+1} - xC,$$

so

$$e^{x^2+1}y' \neq -xy^2.$$
The problem of finding a particular solution of a differential equation satisfying a condition of the form \( y(x_0) = y_0 \) is called an **initial value problem**. The condition \( y(x_0) = y_0 \) is called an **initial condition**.

To solve an initial-value problem, first find the general solution, then use the initial condition \( y(x_0) = y_0 \) to determine a value for the constant.

**Example.** Solve the initial-value problem \( \frac{dy}{dt} = 5 - y \), \( y(3) = 1 \).

To divide by \( 5 - y \), we need \( y \neq 5 \). So we consider two cases.

Case \( y(t) = 5 \): The constant function \( y(t) = 5 \) clearly satisfies \( \frac{dy}{dt} = 5 - y \), but it doesn’t satisfy the initial condition \( y(3) = 1 \).

Case \( y(t) \neq 5 \):

\[
\frac{dy}{5 - y} = dt
\]

\[
\int \frac{dy}{5 - y} = \int dt
\]

\[
- \ln |5 - y| = t + C \quad C \text{ is an arbitrary constant}
\]

\[
\ln |5 - y| = -t - C
\]

\[
|5 - y| = e^{-t-C}
\]

\[
5 - y = \pm e^{-t-C} \quad \text{because } e^{-t-C} > 0
\]

Applying the initial condition \( y(3) = 1 \) gives

\[
5 - 1 = \pm e^{-3-C}
\]

\[
4 = \pm e^{-3-C}
\]

\[
4 = e^{-3-C} \quad \text{we must have the + sign because } e^{-t-C} > 0
\]

\[
\ln 4 = -3 - C
\]

\[
C = -3 - \ln 4
\]

Therefore

\[
5 - y(t) = \pm e^{-t-(-3-\ln 4)} = \pm e^{-t+3+\ln 4}.
\]

Since \( y(t) \) is continuous and \( e^{-t-(-3-\ln 4)} > 0 \), we must make the same sign choice as above. Therefore

\[
5 - y(t) = e^{-t+3+\ln 4}
\]

\[
y(t) = 5 - e^{-t+3+\ln 4}
\]
Additional Examples.

Example. Solve \( \frac{dy}{dx} = \frac{8x^3 y}{2y^2 + 1} \)

\[
\int \frac{2y^2 + 1}{y} \, dy = \int 8x^3 \, dx
\]

\[
\int (2y + y^{-1}) \, dy = \int 8x^3 \, dx
\]

\[ y^2 + \ln |y| = 2x^4 + C \]

where \( C \) is an arbitrary constant. In this case, we can’t express \( y \) explicitly as a function of \( x \). The last equation gives the general solution implicitly.

Example. Solve \( \frac{dy}{dt} = 5 - y \).

To divide by \( 5 - y \), we need \( y \neq 5 \). So we consider two cases.

Case \( y(t) \neq 5 \):

\[
\frac{dy}{5 - y} = dt
\]

\[
\int \frac{dy}{5 - y} = \int dt
\]

\[ \ln |5 - y| = t + C \quad C \text{ is an arbitrary constant} \]

\[ |5 - y| = e^{t+C} \]

\[ 5 - y = \pm e^C e^t \]

\[ y(t) = 5 \pm e^C e^t \]

Case \( y(t) = 5 \): The constant function \( y(t) = 5 \) clearly satisfies \( \frac{dy}{dt} = 5 - y \).

The general solution consists of the functions \( y(t) = 5 \), \( y(t) = 5 \pm e^C e^t \) where \( C \) is any real number.

We can write this more compactly. Since \( C \) can be any real number, \( e^C \) can be any positive number. So \( \pm e^C \) can be any non-zero number. Thus we can write the general solution as

\[ y(t) = 5 + Ae^t \]

where \( A \) is any real number.

Example Find a solution of \( y' = 2xe^{-y} + e^{-y} \) that satisfies the initial condition \( y(-1) = \ln 2 \).

\[
\frac{dy}{dx} = (2x + 1)e^{-y}
\]

\[ e^{y} \, dy = (2x + 1) \, dx \]

\[
\int e^{y} \, dy = \int (2x + 1) \, dx
\]

\[ e^{y} = x^2 + x + C \]

Enforcing the initial condition \( y(-1) = \ln 2 \) gives

\[ e^{\ln 2} = (-1)^2 + (-1) + C \]

\[ 2 = C \]
Therefore

\[ e^y = x^2 + x + 2 \]
\[ y(x) = \ln(x^2 + x + 2) \]