Complex Numbers

Define \( i \) to be a number such that \( i^2 = -1 \)

Note: \( i \) cannot be a real number.

The set \( \mathbb{C} \) of complex numbers consists of all numbers of the form \( a + ib \), where \( a, b \) are real.

### Addition/Subtraction/Multiplication/Division

\[ \omega = 1 + 2i, \quad z = 3 + 4i \]

\[ \omega + z = (1 + 2i) + (3 + 4i) = (1 + 3) + i(2 + 4) = 4 + 6i \]

\[ \omega - z = (1 + 2i) - (3 + 4i) = -2 - 2i \]

\[ \omega z = (1 + 2i)(3 + 4i) = 3 + 4i + 6i + 2 \cdot 8i^2 = 3 - 6 + i(4 + 6) \]

\[ = -5 + 10i \quad (\omega z = -5 + 10i) \]

\[ \frac{\omega}{z} = \frac{1 + 2i}{3 + 4i} = \frac{1 + 2i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{(3 + 8) + i(6 - 4)}{(9 + 16) + i(12 - 12)} \]

\[ = \frac{11 + i2}{25} \]

\[ = \frac{11}{25} + \frac{i2}{25} \]

### Terminology

\( z = a + ib \) \( (a, b \text{ real}) \)

We say \( z \) is real if \( b = 0 \)

We say \( z \) is purely imaginary if \( a = 0 \)
Real Part, Imaginary Part, Conjugate, Modulus,

For \( z = a + ib \), define

\[
\operatorname{Re}(z) = a \quad \text{(real part of } z) \]

\[
\operatorname{Im}(z) = b \quad \text{(imaginary part of } z) \]

\[
\bar{z} = a - ib \quad \text{(complex conjugate of } z) \]

\[
|z| = \sqrt{a^2 + b^2} \quad \text{(modulus/absolute value/norm/length of } z) \]

Relationships

\[
\frac{z + \bar{z}}{2} = \frac{a + ib + a - ib}{2} = \frac{2a}{2} = a = \operatorname{Re}(z) \]

\[
\frac{z - \bar{z}}{2i} = \frac{(a + ib) - (a - ib)}{2} = \frac{2ib}{2i} = b = \operatorname{Im}(z) \]

\[
z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2 \]

Division Again

\[
\frac{\omega}{z} = \frac{\omega \bar{z}}{|z|^2} = \frac{\omega \bar{z}}{|z|^2} \]

\[
\omega = 1 + 2i \quad \text{and} \quad z = 3 + 4i \]

\[
\frac{\omega}{z} = \frac{1 + 2i}{|z|^2} = \frac{(1+2i)(3-4i)}{\overline{z}^2 + 4^2} = \frac{(3+8) + i(6-4)}{9+16} \]

\[
= \frac{11 + 2i}{25} = \frac{11}{25} + \frac{2}{25}i \]
\[ C = \mathbb{R}^2 \]

Identify \( z = a + ib \) in \( C \) with \( (a, b) \) in \( \mathbb{R}^2 \)

\[ r = |z| = \sqrt{a^2 + b^2} \]

\[ \Theta = \text{argument of } z \]

\[ a = r \cos \Theta, \quad b = r \sin \Theta \]

\[ z = a + ib = r \cos \Theta + i r \sin \Theta = r \left( \cos \Theta + i \sin \Theta \right) \]

**Euler's Formula**

\[ e^{i \Theta} = \cos \Theta + i \sin \Theta \]

**Proof:** Try it / ask me again after you learn about power series in Math 101.

\[ e^{i \Theta} = \sum_{n=0}^{\infty} \frac{(i \Theta)^n}{n!} = \sum_{n \text{ odd}} \frac{(i \Theta)^n}{n!} + \sum_{n \text{ even}} \frac{(i \Theta)^n}{n!} \]

\[ = \sum_{k=0}^{\infty} \frac{(i \Theta)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i \Theta)^{2k+1}}{(2k+1)!} \]

\[ = \sum_{k=0}^{\infty} \frac{(i)^{2k}}{(2k)!} \Theta^{2k} + i \sum_{k=0}^{\infty} \frac{(i)^{2k}}{(2k+1)!} \Theta^{2k+1} \]

\[ = \cos(\Theta) + i \sin(\Theta) \]

\[ i^{2k} = (i^2)^k = (-1)^k \]

\( (i^3 = i^2 \cdot i = -1 \cdot i = -i) \)
Note: \( |e^{i\theta}| = \cos^2 \theta + \sin^2 \theta = 1 \)

\[ i = e^{i\pi/2} \]
\[ 1 = e^{i0} = e^{i2\pi} \]
\[ -1 = e^{i\pi} \]
\[ 1 = e^{i0} = e^{i2\pi} \]
\[ e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1 + i0 = 1 \]

**Polar Form**

The polar form of \( z = a + ib \) is

\[ z = r e^{i\theta} \]

\[ r = |z|, \quad \theta = \text{argument of } z \]

Note: \( z = re^{i\theta} = re^{i\theta} e^{i2\pi} = re^{i(\theta + 2\pi)} \)

**Example**

\[ z = \sqrt{3} + i3 \]
\[ |z| = \sqrt{(\sqrt{3})^2 + 3^2} = \sqrt{3 + 9} = \sqrt{12} = 2\sqrt{3} \]

\[ \sin \theta = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \]
\[ \cos \theta = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} \]

\[ \therefore \theta = \frac{\pi}{3} \]

\[ z = \sqrt{3} + i3 = 2\sqrt{3} e^{i\pi/3} \]

Can also write \( z = 2\sqrt{3} e^{i(\pi/3 + 10\pi)} = 2\sqrt{3} e^{i(\pi/3 - 4\pi)} = \ldots \)
Multiplication/Division

\[ z_1 = r_1 e^{i \theta_1}, \quad z_2 = r_2 e^{i \theta_2} \]

\[ z_1 z_2 = r_1 r_2 e^{i (\theta_1 + \theta_2)} \]  
\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i (\theta_1 - \theta_2)} \]

(e^a e^b = e^{a+b})

Roots of Polynomials

Example

Find the roots of \( 3x^2 + 4x + 5 \)  
(i.e., find the solutions of \( 3x^2 + 4x + 5 = 0 \))

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4(3)(5)}}{2(3)} \]

\[ = \frac{-4 \pm \sqrt{-44}}{6} = \frac{-4 \pm i \sqrt{44}}{6} \]

\[ = \frac{-4 \pm i 2 \sqrt{11}}{6} = \frac{-2 \pm i \sqrt{11}}{3} \]

The roots are \( \frac{-2 + i \sqrt{11}}{3} \) and \( \frac{-2 - i \sqrt{11}}{3} \)

Note this polynomial has no real roots, but 2 complex roots.

Fundamental Theorem of Algebra

The polynomial

\[ p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \]

with coefficients \( a_0, \ldots, a_n \) in \( \mathbb{C} \) (which includes the case \( a_0, \ldots, a_n \) in \( \mathbb{R} \)) always has exactly \( n \) roots \( \lambda_1, \ldots, \lambda_n \) in \( \mathbb{C} \) (some of which may be repeated and some of which may be real) and

\[ p(z) = a_n(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n) \]
Example. Repeated Roots.

\[ x^2 + 2x + 1 = (x+1)(x+1) \]
Roots are \( \lambda_1 = -1 \), \( \lambda_2 = -1 \).

Finding \( n \)th Roots

\( w \) is called an \( n \)th root of \( z \) if \( w^n = z \) \((w^n - z = 0)\)

Every \( z \) in \( \mathbb{C} \) has \( n \) \( n \)th roots in \( \mathbb{C} \)

If \( z = r e^{i\theta} \), then

\[ r^{1/n} e^{i\theta/n} \]

is always an \( n \)th root of \( z \).
Here \( r^{1/n} = \text{unique positive } n \text{th root of } r = |z| \)

The set of all \( n \)th roots of \( z \) is

\[ r^{1/n} e^{i\theta/n} e^{i2\pi k/n}, \quad k = 0, 1, \ldots, n-1 \]

\[ r^{1/n} e^{i(\theta/n + 2\pi k/n)} \]

Check: If \( z = r e^{i\theta} \), \( w = r^{1/n} e^{i\theta/n} e^{i2\pi k/n} \), then

\[ w^n = (r^{1/n} e^{i\theta/n} e^{i2\pi k/n})^n = r e^{i\theta} e^{i2\pi k} = r e^{i\theta} \]

because \( e^{i2\pi k} = (e^{i2\pi})^k = 1 \) \( \text{for } k = 1 \)

Example

Find the 3rd roots of \( z = -27 \)

\[ r = |z| = 27, \quad r^{1/3} = 3 \]

\(-1 = e^{i\pi}, \quad \therefore \theta = \pi \)

\[ \therefore z = -27 = r e^{i\theta} = 27 e^{i\pi} \]

The 3rd roots are

\[ 3 e^{i\pi/3}, \quad 3 e^{i\pi/3} e^{i2\pi/3}, \quad 3 e^{i\pi/3} e^{i4\pi/3} \]

\[ = 3 e^{i\pi} = -3 \quad \text{and} \quad = 3 e^{i5\pi/3} \]
Matrices, vectors, scalars are complex from now on.