Linearity of Expectation

\[ E[X_1 + \ldots + X_n] = E[X_1] + \ldots + E[X_n] \]

Indicator Random Variable of Event \( A \)

\[ I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur} \end{cases} \]

\[ E[I_A] = P(A) \]

Example

Draw 5 cards from a standard deck of 52.

\( X = \) number of aces drawn.

Find \( E[X] \)

Four aces in the deck. Label them 1, 2, 3, 4.

\[ I_i = \begin{cases} 1 & \text{if ace } i \text{ was drawn} \\ 0 & \text{otherwise} \end{cases} \]

\[ X = I_1 + I_2 + I_3 + I_4 \]


\[ E[I_i] = P(\text{ace } i \text{ was drawn}) \]

\[ = \frac{\#A}{\#D} = \frac{\#\text{combinations of 5 cards including } i}{\#\text{combinations of 5 cards}} \]

\[ = \frac{\binom{1}{1} \cdot \binom{51}{4}}{\binom{52}{5}} = \frac{51 \cdot 50 \cdot 49 \cdot 48}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \]

\[ = \frac{5}{52} \]
\[ E[X] = \frac{4.5}{5^2} = \frac{9}{13} \]

**Example**

The sequence of coin flips

\[ HHTHHHTTTTHTHHHT \]

has 2 runs of heads of length 3, 1 run of length 2, and 1 run of length 1.

Consider a coin where \( P(\text{heads}) = p \).

If the coin is flipped 100 times, what is the expected number of runs of heads of length 3?

\[ X = \text{number of runs of heads of length 3} \]

Want \( E[X] \)

For \( i \in \{1, \ldots, 98\} \),

\[ I_i = \begin{cases} 1 & \text{if a run of 3 heads starts at flip } i \\ 0 & \text{otherwise} \end{cases} \]

\[ X = \sum_{i=1}^{98} I_i \]

\[ E[X] = \sum_{i=1}^{98} E[I_i] \]

\[ E[I_i] = P(\text{run of 3 heads starts at flip } i) \]

Run of heads of length 3 starts at flip \( i = 4 \)
only if heads on flips 1, 2, 3 and tails on flip 4.

Run of heads of length 3 starts at flip \( i = 98 \)
only if tails on flip 97 and heads on flips 98, 99, 100.
Run of heads of length 3 starts at flip $i \in \{2, \ldots, 97\}$ only if heads on flips $i$, $i+1$, $i+2$ and tails on flips $i-1$ and $i+3$

$E[I_1] = P(\text{run of 3 heads starts at flip } 1) = p^3(1-p)$

$E[I_{98}] = P(\text{run of 3 heads starts at flip } 98) = (1-p)p^3$

For $i = 2, \ldots, 97$,

$E[I_i] = P(\text{run of 3 heads starts at flip } i) = (1-p)p^3 \cdot (1-p) = p^3(1-p)^2$

$E[C] = \sum_{i=1}^{98} E[I_i] = 2p^3(1-p) + 96p^3(1-p)^2$