Bernoulli Process (Coin Flip Process)
Infinite sequence of independent Ber(p) trials (at discrete time points)
- # of successes in n trials \(\sim \text{Bin}(n, p)\)
- # of trials until k\textsuperscript{th} success \(\sim \text{NegBin}(k, p)\)
- # of trials between k\textsuperscript{th} success and (k+1)\textsuperscript{th} success \(\sim \text{Geom}(p)\)

Poisson Process (Continuous Time Analog of Bernoulli Process)
Sequence of events occurring randomly over a continuous period of time, starting at time 0.

Example: Arrivals at post office.
- Calls at a call center.
- Requests to a computer server.
- Decays of radioactive atoms.

Notation:
\[ |I| = \text{length of time interval } I \]
Example: If \( I = [a, b] \), then \( |I| = b - a \).

\( N(I) = \# \text{ of occurrences in time interval } I \)

Properties that define a Poisson Process with rate \( \lambda \)

1. \( N(I) \sim \text{Poisson}(\lambda |I|) \) for any time interval \( I \subset [0, \infty) \)
(2) If \( I_1, \ldots, I_n \) are disjoint intervals, then
\[ N(I_1), \ldots, N(I_n) \] are independent.

Note: \( \lambda = \frac{E(N(I))}{|I|} \) = average number of
events per unit time
in any interval \( I \).

Recall: \( N(I) \sim \text{Poisson}(\lambda |I|) \) means
\[ P(N(I) = k) = e^{-\lambda |I|} \left( \frac{\lambda |I|}{k!} \right)^k \text{ for } k = 0, 1, 2, \ldots \]

**Example**

\( T_1 = \text{time until 1st occurrence}. \)
\( T_1 \sim \text{Exp}(\lambda). \)

Prove i).

we find the cdf of \( T_1 : P(T_1 \leq t) \)

If \( t < 0 \), then \( P(T_1 \leq t) = 0 \)

If \( t \geq 0 \), then
\[ \{T_1 > t\} = \{1^{st} occurrence is after time t\} \]
\[ = \{zero occurrences at or before time t\} \]
\[ = \{N([0,t]) = 0\} \]
\[ P(T_1 > t) = P(N([0,t]) = 0) = e^{-\lambda t} \times 0^0 = e^{-\lambda t} \]
\[ P(T_1 \leq t) = \begin{cases} 1-e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \]
This is exactly the cdf of $\text{Exp}(\lambda)$.
\[ \therefore T_i \sim \text{Exp}(\lambda). \]

Example
\[ T_k = \text{time until the k^{th} occurrence} \]
Find the pdf of $T_k$

We'll first find the cdf $F$
Then find the the pdf $f$ using
\[ f(t) = \frac{d}{dt} F(t) \]

If $t < 0$, then $F(t) = P(T_k \leq t) = 0$
If $t \geq 0$, then
\[ \{ T_k > t \} = \{ k^{th} occurrence is after time t \} \]
\[ = \{ at most k-1 occurrences at or before time t \} \]
\[ = \{ N(t0, t1) \leq k-1 \} \]

\[ \therefore P(T_k > t) = P(N(t0, t1) \leq k-1) \]
\[ = \sum_{l=0}^{k-1} P(N(t0, t1) = l) \]
\[ = \sum_{l=0}^{k-1} e^{-\lambda t} (\lambda t)^l \]

\[ \therefore F(t) = P(T_k \leq t) = \begin{cases} 1 - \sum_{l=0}^{k-1} \frac{e^{-\lambda t} (\lambda t)^l}{l!} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \]
If \( t + c < 0 \), then \( f(t) = \frac{d}{dt} F(t) = 0 \)

If \( t + c \geq 0 \), then

\[
f(t) = \frac{d}{dt} F(t) = -\sum_{l=0}^{k-1} \frac{\lambda^l}{l!} d e^{-\lambda t} t^l
\]

\[
= -\sum_{l=0}^{k-1} \frac{\lambda^l}{l!} \left(-\lambda e^{-\lambda t} t + e^{-\lambda t} t^l - \left(\lambda t^l + e^{-\lambda t} t^l d\right)\right)
\]

\[
= \lambda e^{-\lambda t} \sum_{l=0}^{k-1} \frac{\lambda^l}{l!} \left(\lambda t^l - e^{-\lambda t} t^l - \left(\lambda t^l + e^{-\lambda t} t^l d\right)\right)
\]

\[
= \lambda e^{-\lambda t} \left(\frac{k-1}{l=0} \frac{\lambda^l t^l}{l!} - e^{-\lambda t} \sum_{l=0}^{k-1} \frac{\lambda^l t^l}{l!} d\right)
\]

\[
= \lambda e^{-\lambda t} \left(\frac{k-1}{l=0} \frac{\lambda^l t^l}{l!} - \left(\sum_{l=0}^{k-1} \frac{\lambda^l t^l}{l!} - \sum_{l=0}^{k-1} \frac{\lambda^l t^l}{l!} d\right)\right)
\]

\[
= \lambda e^{-\lambda t} \left(\frac{k-1}{l=0} \frac{\lambda^l t^l}{l!} - \sum_{l=0}^{k-1} \frac{\lambda^l t^l}{l!} \frac{d}{(l-1)!}\right)
\]

\[
= \lambda e^{-\lambda t} \left(\frac{k-1}{l=0} \frac{\lambda^l t^l}{l!} - \frac{k-2}{l=0} \frac{\lambda^l t^l}{l!}\right)
\]

\[
= \lambda e^{-\lambda t} \left(\frac{k-1}{l=0} \frac{\lambda^l t^l}{(l-1)!}\right)
\]

\[
= e^{-\lambda t} \frac{\lambda^k + k-1}{(k-1)!}
\]
\[ f(x) = \begin{cases} 
\frac{e^{-x} x^{k-1}}{(k-1)!} & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases} \]

**Definition: Gamma Distribution**

Let \( x > 0, k \in \mathbb{N} \).

\( X \) has a gamma distribution with parameters \( \alpha \) and \( k \) if its pdf is

\[ f(x) = \begin{cases} 
\frac{e^{-x} x^{k-1}}{(k-1)!} & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases} \]

Write \( X \sim \text{Gamma}(\alpha, k) \)

**Note:** If \( k=1 \), \( \text{Gamma}(\alpha, k) = \text{Exp}(\alpha) \)

The last example says

\[ \text{k^\text{th} time until k^\text{th} occurrence } \sim \text{Gamma}(\alpha, k) \]
Exercise (Solution at End of Lecture)
The Moment Generating Function of Gamma(α, k)

\[ M(t) = \left( \frac{\lambda}{\lambda-t} \right)^k \quad \text{for} \; t < \lambda. \]

Example
If \( X, Y \) are independent with \( X \sim \text{Gamma}(\alpha, k) \) and \( Y \sim \text{Gamma}(\alpha, l) \), then
\( X + Y \sim \text{Gamma}(\alpha, k+l) \).

\[
M_{X+Y}(t) = M_X(t) M_Y(t) \\
= \left( \frac{\lambda}{\lambda-t} \right)^k \left( \frac{\lambda}{\lambda-t} \right)^l \\
= \left( \frac{\lambda}{\lambda-t} \right)^{k+l} \quad (t < \lambda)
\]

This is the MGF of a \( \text{Gamma}(\alpha, k+l) \) random variable
\[
\therefore X + Y \sim \text{Gamma}(\alpha, k+l)
\]

Generalization
If \( X_1, \ldots, X_n \) independent with \( X_i \sim \text{Gamma}(\alpha, k_i) \), then
\( X_1 + \ldots + X_n \sim \text{Gamma}(\alpha, k) \)
where \( k = k_1 + \ldots + k_n \)
**Special Case**
If $X_1, ..., X_n$ independent with $X_i \sim \text{Exp}(\lambda)$
then
$$X_1 + \ldots + X_n \sim \text{Gamma}(\lambda, n)$$

**Hint:** $\text{Exp}(\lambda) = \text{Gamma}(\lambda, 1)$

**Example**
Consider a Poisson process with rate $\lambda$.
$T_k =$ time until $k^{th}$ occurrence

Define $X_1 = T_1 =$ time until $1^{st}$ occurrence.
$\therefore X_1 \sim \text{Exp}(\lambda)$

Define $X_k = T_k - T_{k-1} =$ time between $(k-1)^{th}$ occurrence and $k^{th}$ occurrence

$$X_1 + \ldots + X_k = T_k \sim \text{Gamma}(\lambda, k)$$

Guess:
$$X_1, ..., X_k \sim \text{Exp}(\lambda) \text{ independent}$$

Guess is correct.
**Informal justification:**
We know $X_1 = T_1 =$ time of $1^{st}$ occurrence $\sim \text{Exp}(\lambda)$
Consider $X_2 = T_2 - T_1 =$ time from $1^{st}$ occurrence to $2^{nd}$ occurrence

Since disjoint time intervals in a Poisson process are independent, the past of the process
(up to time $T_1$) is independent of the future (after time $T_2$).
Starting from time $T_1$, the future is like starting a fresh Poisson process.

Therefore

$$X_2 = T_2 - T_1 = \text{time from 1st occurrence until 2nd occurrence}$$

has the same distribution as

$$X_1 = T_1 = \text{time from start until 1st occurrence}$$

and $X_1, X_2$ independent.

Therefore

$$X_2 = T_2 - T_1 = \text{d} X_1 = T_1 \sim \text{Exp} (\lambda)$$

and $X_1, X_2$ independent.

\[ \text{equal in distribution.} \]
Variables Associated with Poisson Process of Rate $\lambda$

- # of occurrences in time interval $I \sim \text{Poisson}(\lambda|I|)$
- time until $k^{th}$ occurrence $\sim \text{Gamma}(\lambda, k)$
- time between $k^{th}$ and $(k+1)^{th}$ occurrence $\sim \text{Exp}(\lambda)$

Variables Associated with Bernoulli Process of Probability $p$

(Infinte Sequence of independent Bernoulli trials)

- # of occurrences in $n$ trials $\sim \text{Bin}(n, p)$
- # of trials until $k^{th}$ success $\sim \text{Negbin}(k, p)$
- # of trials between $k^{th}$ and $(k+1)^{th}$ success $\sim \text{Geom}(p)$
Example

Calls come into a call center according to a Poisson process at rate 20/hr, starting at midnight $t=0$.

(a) Find the probability that there are no calls between 9 am and 10 am, but 60 calls between 10 am and noon.

$\lambda = 20$

The probability is

$$P(N([9,10]) = 0 \text{ and } N([10,12]) = 60)$$

$$= P(N([9,10]) = 0) P(N([10,12]) = 60)$$

(by independence on disjoint intervals)

$$= \frac{e^{-20 \cdot 1} \cdot (20 \cdot 1)^0}{0!} + \frac{e^{-20 \cdot 2} \cdot (20 \cdot 2)^{60}}{60!}$$

$$\approx 0.000678651281\ldots$$

(b) Find the probability that the 99th calls arrives after 3 am and the 100th call arrives more than 1 hour after 99th.

$\lambda = 20$

$T_{99} = \text{time of 99th call } \sim \text{Gamma}(20, 99)$

$W_{100} = \text{time between 99th and 100th call } \sim \text{Exp}(20)$

Want $P(T_{99} > 3, W_{100} > 1)$
\[ T_{qq} = W_1 + \ldots + W_{qq} \]
\[ W_i = \text{time between } (i-1)\text{th and } i\text{th call} \]
\[ W_1, \ldots, W_{100} \text{ indep. Exp}(20) \]

\[ \therefore T_{qq} = W_1 + \ldots + W_{qq} \text{ and } W_{100} \text{ indep.} \]

\[ \because P(T_{qq} > 3, W_{100} > 1) = P(T_{qq} > 3)P(W_{100} > 1) \]

\[ P(W_{100} > 1) = e^{-20.1} \]

\[ P(T_{qq} > 3) = \sum_{k=3}^{\infty} \frac{e^{-20} \cdot 20^{k-1} \cdot k!}{(k-1)!} dx = \sum_{k=3}^{\infty} \frac{e^{-20} \cdot 20^k}{20} \cdot \frac{99!}{98!} \]

\[ = \text{hard (integrate by parts a lot)} \]

\[ P(T_{qq} > 3) = P(N([0,3]) \leq 98) \]

\[ = \sum_{l=0}^{98} P(N([0,3]) = l) \]

\[ = \sum_{l=0}^{98} e^{-20.3} \left( \frac{20.3}{l!} \right)^l \]

\[ \therefore P(T_{qq} > 3, W_{100} > 1) = \left( \sum_{l=0}^{98} e^{-20.3} \left( \frac{20.3}{l!} \right)^l \right) \cdot e^{-20.1} \]
We can define the Gamma distribution with the parameter $k \in \mathbb{N}$ replaced by $r > 0$.

**Definition: Gamma Distribution**

Let $x > 0$, $r > 0$.

$X \sim \text{Gamma}(x, r)$ if $X$ has pdf

$$f(x) = \begin{cases} \frac{e^{-x} x^r}{\Gamma(r)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$
\Gamma(r)$ is defined so that $\int_{-\infty}^{\infty} f(x) \, dx = 1$

\[ \therefore \Gamma(r) = \int_{0}^{\infty} e^{-x} x^{r-1} \, dx \]

If $r \in \mathbb{N}$, then $\Gamma(r) = r!$.
Exercise:
The MGF of Gamma(\(\lambda, \tau\)) is

\[ M(t) = \left( \frac{\lambda}{\lambda-t} \right)^\tau \text{ for } t < \lambda \]

Solution:

\[ M(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} f(x)dx = \int_{0}^{\infty} e^{xt} \frac{e^{-\lambda x} x^{\tau-1}}{\Gamma(\tau)} dx \]

\[ = \int_{0}^{\infty} \frac{e^{-(\lambda-t)x} x^{\tau-1}}{\Gamma(\tau)} dx \]

\[ = \left( \frac{\lambda}{\lambda-t} \right)^\tau \int_{0}^{\infty} \frac{e^{-(\lambda-t)(\lambda-t)} x^{\tau-1}}{\Gamma(\tau)} dx \]

\[ = \left( \frac{\lambda}{\lambda-t} \right)^\tau \text{ for } t < \lambda \]

because the last integral is the integral of the pdf of a Gamma(\(\lambda-t, \tau\)) random variable over \((-\infty, \infty)\).