Independence and Joint Distributions

\( X_1, \ldots, X_n \) are independent IFF
\[
P(X_1 \in B_1, \ldots, X_n \in B_n) = P(X_1 \in B_1) \cdots P(X_n \in B_n)
\]
for all sets \( B_1, \ldots, B_n \subseteq \mathbb{R} \)

Equivalently
\[
P(X_1 = t_1, \ldots, X_n = t_n) = P(X_1 = t_1) \cdots P(X_n = t_n)
\]
for all \( t_1, \ldots, t_n \in \mathbb{R} \)

[Special case when \( B_i = (-\infty, t_i] \)]

Equivalently
\[
F_{X_1, \ldots, X_n}(t_1, \ldots, t_n) = F_{X_1}(t_1) \cdots F_{X_n}(t_n)
\]
for all \( t_1, \ldots, t_n \in \mathbb{R} \)

The discrete rvs \( X_1, \ldots, X_n \) are independent IFF the pmf's factor
\[
P_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = P_{X_1}(x_1) \cdots P_{X_n}(x_n)
\]
for all \( x_1, \ldots, x_n \in \mathbb{R} \)

Enough to have \( x_i \in \text{Range}(X_i), \ldots, x_n \in \text{Range}(X_n) \).

The jointly continuous rvs \( X_1, \ldots, X_n \) are independent IFF
\[
f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)
\]
for all \( x_1, \ldots, x_n \in \mathbb{R} \)
Example

\[ X \sim \text{Poisson}(\lambda), \quad Y \sim \text{Bernoulli}(p) \]

\[ X, Y \text{ are independent} \]

Find joint pmf

\[
P_X(k) = \frac{e^{-\lambda} \lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots
\]

\[
P_Y(l) = (1-p)^{l-1} p \quad \text{for } l = 1, 2, 3, \ldots
\]

By independence, the pmf of the pair \( (X, Y) \) evaluated at \( k, l \)

\[
P_{X,Y}(k, l) = P_X(k) P_Y(l) = \frac{e^{-\lambda} \lambda^k}{k!} \cdot (1-p)^{l-1} p
\]

\[ \text{for } k = 0, 1, 2, \ldots ; \quad l = 1, 2, 3, \ldots \]

Example

\( (X, Y) \) is uniformly distributed on the rectangle \( A = [a, b] \times [c, d] = \]

Are \( X, Y \) independent?

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f_{X,Y}(x, y) \, dx \, dy
\]

\[ f_{X,Y}(x, y) = \frac{1}{(b-a)(d-c)} \quad \text{for } a \leq x \leq b, \quad c \leq y \leq d
\]

If \( X \in [a, b] \), then

\[
P_X(x) = \int_{\text{lower limit}}^{\text{upper limit}} f_{X,Y}(x, y) \, dy
\]

\[
P_X(x) = \int_c^d \frac{1}{(b-a)(d-c)} \, dy = \frac{d-c}{b-a}
\]
\[ x \neq \{a, b\}, \quad \text{then} \]
\[ f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{a}^{b} 0 \, dy = 0 \]
\[ f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{if } x \notin [a, b] \end{cases} \]

Then \( f_X(x) f_Y(y) = \begin{cases} \frac{1}{(b-a)^2} & \text{if } x \leq y \leq 1 - x \\ 0 & \text{otherwise} \end{cases} \)

\[ f_X(x) f_Y(y) = f_{X,Y}(x, y) \quad \forall x, y \]

Show that \( X, Y \) are independent

**Example**

\( (X, Y) \) are uniformly distributed on the triangle \( \triangle \) with vertices \((0,0), (1,0), (0,1) \)

\( X, Y \) independent

Find the marginal pdf's of \( X \) and \( Y \) and from that are \( X, Y \) independent?
The joint pdf is
\[ f_{X,Y}(x,y) = \begin{cases} \frac{1}{1-e^{-2}} & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \]

\[
\forall y \in [0,1], \text{ then } f_X(x) = \int_0^x f_{X,Y}(x,y) \, dy = \int_0^x 2 \, dy = ay \bigg|_0^1 = a(1-x)
\]

continued next class...
\[ P_x(u) P_y(v) = \sum_{k=-\infty}^{\infty} P_x(\xi_u - k) P_y(\xi_v - k) \]

We also have:

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_x(u) P_y(v) \, \text{d}u \, \text{d}v = 1 \]

Therefore, if \( y = x^2 \),

we can use another way to determine:

\[ P(x = \frac{y^2}{2}, y > \frac{1}{2}) = 0 \]

\[ P(x > \frac{1}{2}) = \frac{1}{2} \]

\[ P(x > y > \frac{1}{2}) = \text{full \ expression} \]
Preliminary: Let \( f \) be a non-negative \( \mathbb{R} \)-valued function of \( \mathbb{R} \times \mathbb{R} \) such that
\[
\int_{\mathbb{R}^2} f(x_1, x_2) \, dx_1 \, dx_2 < \infty.
\]

Given \( x_1, x_2 \in \mathbb{R} \), we can consider the function of \( x_1 \)
\[
\rho(x_1) = \frac{g_2(x_2)}{\int_{-\infty}^{\infty} g_1(x_1) \, dx_1}.
\]

Then for any \( x_1, x_2 \in \mathbb{R} \), we have
\[
f_{X_2}(x_2) = g_2(x_2) \rho(x_1) \, dx_1.
\]
Example

With joint pdf

\[
(f(x,y), \frac{1}{x^2} y \text{ if } 1 < x + z \text{ and } y > 0)
\]

Given that \( f(x,y) \) may be found for all \( x, y \)

\[
g(x) = \int_{-\infty}^{\infty} f(x,y) \, dy
\]

\[
u = 0 \text{ or other case}
\]

\[
\text{expected }
\]

Fact about expectation (Proof: Exercise)

Expected value of the conditional

\[
E[X|H] = \int_{-\infty}^{\infty} x f(x|h) \, dx
\]

\[
h(x) = \int_{-\infty}^{\infty} f(x,y) \, dy
\]

Example: the \( X, Y, Z \) are independent

\[
x \quad y + z
\]