Law of Large Numbers

Given \( X_1, \ldots, X_n \) indep with same distribution and \( E(X_i) = \mu \).

For any \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right| < \varepsilon \right) = 1
\]
equivalently
\[
\lim_{n \to \infty} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right| \geq \varepsilon \right) = 0
\]

The probability that the sample mean \( \frac{1}{n} \sum_{i=1}^{n} X_i \) differs from the true mean \( \mu \) by more than \( \varepsilon \) becomes extremely small as \( n \) grows.

\[
\text{sample mean } \xrightarrow{n \to \infty} \text{true mean}
\]

Proof: Later in the course.

Law of Large Numbers: Binomial

\( S_n \sim \text{Bin}(n, p) \)

\( S_n = \sum_{i=1}^{n} X_i \) with \( X_1, \ldots, X_n \) indep \( \text{Ber}(p) \)

\[
\Rightarrow \frac{S_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i
\]
Idea: $\frac{S_n}{n} = \text{estimate for } p$ where $p = \text{probability of heads}$

For any $\varepsilon > 0$,
$$\lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - p \right| < \varepsilon \right) = 1$$

equivalently
$$\lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - p \right| \geq \varepsilon \right) = 0$$

The estimate converges to the true value as the number of flips goes to $\infty$. 