Sets
A set is a collection of objects, usually numbers. The objects in the set are called the elements of the set.

Example
\[ A = \{1, 2, 3\} \]
\[ 1 \in A \quad (1 \text{ is an element of } A) \]
\[ 4 \notin A \quad (4 \text{ is not an element of } A) \]

Definitions
\[ \mathbb{N} = \text{set of positive integers} = \{1, 2, 3, \ldots\} \]
\[ \mathbb{Z} = \text{set of integers} = \mathbb{N} \cup \{0, -1, 2, -2, \ldots\} \]
\[ \mathbb{R} = \text{set of real numbers} \]
\[ \emptyset = \text{empty set} = \text{set of no elements} = \varnothing \]

Another Way to Write Sets
\[ \{x \mid x \text{ satisfies } P\} = \text{set of all elements having property } P \]

Examples
\[ [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\} \]
\[ A = \{n \in \mathbb{N} : n \text{ is a square}\} = \{n^2 : n \in \mathbb{N}\} = \{1, 4, 9, 16, 25, \ldots\} \]
**Subsets**

A is a subset of B if every element of A is an element of B. Notation: $A \subseteq B$

$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

if and only if A and B have the same elements

Example

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4, 5, 6\}$

$C = \{7, 8\}$

$A \subseteq B$

$A \not\subseteq C$

$B \not\subseteq A$

**Union, Intersection**

$A \cup B =$ union of A and B

$= \{x : x \in A \text{ or } x \in B\} $

$A \cap B =$ intersection of A and B

$= \{x : x \in A \text{ and } x \in B\} $
\[ A = \{1, 2, 3\} \]
\[ B = \{3, 4\} \]

\[ A \cup B = \{1, 2, 3, 4\} \]
\[ A \cap B = \{3\} \]

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**Union and Intersection of Many Sets**

Let \( A_1, A_2, \ldots \) be sets.

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup \ldots \cup A_n = \{ x : x \in A_i \text{ for some } i \in \{1, \ldots, n\} \} \]

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap \ldots \cap A_n = \{ x : x \in A_i \text{ for all } i \in \{1, \ldots, n\} \} \]
Example
\[ A_1 = \{1\}, \ A_2 = \{1, 2\}, \ A_3 = \{1, 2, 3\}, \ldots \]

\[ \bigcup_{i=1}^{n} A_i = \{1\}, \ldots, n \}
\[ \bigcup_{i=1}^{\infty} A_i = \{1, 2, 3, \ldots \} = \mathbb{N} \]
\[ \bigcap_{i=1}^{5} A_i = \{1\} \]
\[ \bigcap_{i=5}^{10} A_i = A_5 \cap A_6 \cap \ldots \cap A_{10} = \{1, 2, 3, 4, 5\} \]

**Disjoint sets**

\( A \) and \( B \) are called **disjoint** (mutually exclusive) if \( A \cap B = \emptyset \)

\( A_1, A_2, \ldots \) are called disjoint (mutually exclusive) if \( A_i \cap A_j = \emptyset \) for each pair \( i, j \) with \( i \neq j \).

Example
\[ A = \{1, 2, 3\}, \ B = \{7, 22, 45\} \]
\[ A \cap B = \emptyset. \ A, B \text{ are disjoint} \]
\[ A_1 = \{1\}, \ A_2 = \{2\}, \ A_3 = \{3\}, \ldots \]
\[ A_1, A_2, \ldots \text{ are disjoint.} \]
Universe, Complement, Difference

Let \( \Omega \) be a universe set; i.e., a set that contains all the objects of interest in a particular context.

Let \( A, B \) be subsets of \( \Omega \).

Complement: \( A^c = \{ x \in \Omega : x \notin A \} \)

Difference: \( A \setminus B = A - B = \{ x \in \Omega : x \in A \text{ and } x \notin B \} \)

= \( A \cap B^c \)

Example

\( \Omega = \mathbb{R} \)

\( A = [0,1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \)

\( B = \mathbb{Z} \)

\( A^c = (-\infty, 0) \cup (1, \infty) \)

\( A \setminus B = (0,1) \)
Pop Quiz Hot Shot

\[ \Omega^c = \emptyset \]
\[ \emptyset^c = \Omega \]

Pop Quiz Hot Shot

What is \( \bigcup_{n \in \mathbb{Z}} (n, n+1) \)?

Answer: \( \{ x \in \mathbb{R} : x \notin \mathbb{Z} \} = \mathbb{R} \setminus \mathbb{Z} \)

Algebra of Sets

\[(A^c)^c = A\]
\[A \cap A^c = \emptyset\]
\[A \cup \Omega = \Omega\]
\[A \cap \Omega = A\]

De Morgan's Laws

\[(A \cup B)^c = A^c \cap B^c\]
\[\left( \bigcup_{i} A_i \right)^c = \bigcap_{i} A_i^c\]

\[(A \cap B)^c = A^c \cup B^c\]
\[\left( \bigcap_{i} A_i \right)^c = \bigcup_{i} A_i^c\]

Proof:

\[ x \in (A \cup B)^c \iff x \notin A \cup B \iff x \notin A \text{ and } x \notin B \]
\[ \iff x \in A^c \cap B^c \quad \text{(reverse more obvious)} \]

("iff" means "if and only if" means "is equivalent to")

\[ x \in (A \cap B)^c \iff x \notin A \cap B \iff x \notin A \text{ or } x \notin B \]
\[ \iff x \in A^c \cup B^c \]

Rest of Proof: Exercise
Another Useful Identity

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

Proof: Exercise.
**Probability Models**

A probability model is a mathematical description of an uncertain situation. It has two parts:

1. **The sample space** $\Omega = \text{set of all possible outcomes of an experiment}
   
   - Elements of $\Omega$ are outcomes (also called sample points). They are denoted by $w$.
   
   - Subsets of $\Omega$ are called events
   
   - The set of all events is denoted by $F$

2. **The probability measure** $P$
   
   - $P$ is a function from $F$ to $\mathbb{R}$
   
   - To each event $A$, it assigns a number $P(A)$ called the probability of $A$.
   
   - $P(A)$ represents our knowledge or belief about the collective likelihood of the elements of $A$.
   
   - The probability measure $P$ is sometimes called a probability law or probability distribution.