Random Variables

Given a probability model \((\Omega, P)\), a random variable is a function \(X\) going from \(\Omega\) to the set of real numbers \(\mathbb{R}\):

\[
X: \Omega \to \mathbb{R}
\]

To each outcome \(\omega \in \Omega\), \(X\) assigns a real number \(X(\omega)\) called an experimental value or a realization of \(X\).

Remarks:
1. A rv is a function nota variable.
2. Use capitals like \(X, Y, Z\) etc. for rv's. (calc. uses \(f, g, h\), etc.)

Notation:
- \(\mathbb{E}[X] = c^3 = \mathbb{E}[\omega \in \Omega: X(\omega) = c^3]\) if function \(X\) evaluated at \(\omega\) is constant.
- \(\mathbb{E}[a \leq X \leq b] = \mathbb{E}[\omega \in \Omega: a \leq X(\omega) \leq b]\) if event that \(a \leq X \leq b\) holds.
- \(\mathbb{E}[X < c] = \mathbb{E}[\omega \in \Omega: X(\omega) < c]\) if event that \(X < c\) holds.
- \(\mathbb{E}[X \in B] = \mathbb{E}[\omega \in \Omega: X(\omega) \in B]\) if event that \(X \in B\) holds.

Sets of \(\mathbb{E}[X = a, X = b^3, 2X = a^3, 2X = b^3, 2X = a^3, 2X = b^3]\) could be any subset of \(\mathbb{R}\), \(\{X(\omega) = \omega \in \Omega\}\).

Chemically, the definition of a rv requires that all sets of this form \(\mathbb{E}[X = a, b^3, 2X = a^3, 2X = b^3]\) be events.

But usually using the simulation (chapter) that all variables are events.

Section 1.7 * called point events.
Example 1

Experiment: Roll 2 fair 6-sided dice

\[ X \]: outcome for 1st die
\[ X_1 \]: outcome of 2nd die
\[ X \]: sum of outcomes of the 2 dice
\[ Y \]: outcome of 2nd die raised to 5th power

If we write the sample space as

\[ \Omega = \{(i,j) : i, j \in \{1, 2, 3, 4, 5, 6\}\} \]

Then \( X_1 ((i,j)) = i \)
\( X_2 ((i,j)) = j \)
\( X((i,j)) = i + j = X_1 ((i,j)) + X_2 ((i,j)) \)
\( Y((i,j)) = j^5 = X_2 ((i,j))^5 \)

\[ P(X_1 = 3) = P(\exists X_2 = 3) = P(\exists (i,j) \in \Omega : X_1 (i,j) = 3) \]
\[ = P(\exists (i,j) \in \Omega : i = 3) = \frac{1}{6} \]

\[ P(X_1 = 3, X_2 = 5) = P(3 \times 5 \times X_2 = 5) \]
\[ = P(\exists (i,j) \in \Omega : (i,j) = (3,5)) = \frac{1}{36} \]

\[ P(X_1 = 3 \text{ or } X_2 = 5) = P(\exists X_1 = 3 \cap \exists X_2 = 5) \]
\[ = P(X_1 = 3) + P(X_2 = 5) - P(X_1 = 3, X_2 = 5) \]

\[ \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36} \]

by inclusion-exclusion

\[ P(X = 3) = P(X_1 = 1, X_2 = 2, 3) = P(X_1 = 2, X_2 = 1, 3) \]
\[ = P(X_1 = 1, X_2 = 2) + P(X_1 = 2, X_2 = 1) \]
\[ = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18} \]
Example 2
In an experiment involving the transmission of a message:
- time needed to transmit the message
- delay after which the message is received
- the number of symbols received in error are all examples of random variables (rv's)

Example 3
Experiment: Randomly select a person from the population
- the person's height
- the person's weight
- the person's sex

Discrete Random Variables
A discrete random variable is a random variable whose countable (finite or countably infinite).

In Example 1 → all rvs were discrete (due to 2 + 3 = 5 only 3rd rv is discrete

Geometric Series converges to \( \frac{a}{1-r} \)

\[
\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}
\]
Discrete Random Variables (countably infinite)

A discrete rv is a rv whose range is countable.

Recall: The range of a function is the set of possible outputs of the function.

In Example 1, all rv's are discrete. In Example 4 & 3 only the third rv is discrete.

Example 4

Flip a fair coin infinitely many times. The Sample Space consists of all infinite sequences of heads & tails (Hs and Ts).

Let X be the number of flips until first T

Y = 1 if first flip is heads
Y = 0 if " " tails

X = (HHHTTTTHHHTTH) = 4
Y = (HHHTTHHHTTH) = 1

X and Y are discrete random variables.

Range (X) = \{1, 2, 3, 4\} \in \mathbb{N}

Range (Y) = \{0, 1\}