Multiplication Rule (Chain Rule)

\[ P(A_1 \cap A_2) = P(A_1)P(A_2 | A_1) \]

Proof:
\[
\frac{P(A_1 \cap A_2)}{P(A_1)} = P(A_2 | A_1)
\]

The general version of this rule:

\[ P \left( \bigcap_{i=1}^{n} A_i \right) = P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2) \cdots \frac{P(A_n | \bigcap_{i=1}^{n-1} A_i)}{P(A_n | \bigcap_{i=1}^{n-1} A_i)} \]

Proof: The right-hand side is
\[
\frac{P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2)}{P(A)} \cdot \frac{P(A_3 | A_1 \cap A_2)}{P(A_3 | A_1 \cap A_2)} \cdots \frac{P(A_n | \bigcap_{i=1}^{n-1} A_i)}{P(A_n | \bigcap_{i=1}^{n-1} A_i)}
\]

\[ = P(A_1 \cap A_2) \]
**Multiplication Rule**

- \( P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 | A_1) \)
- \( P(\cap_{i=1}^{n} A_i) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdots P(A_n | \cap_{i=1}^{n-1} A_i) \)

**Example 1**

Draw 3 cards from a standard deck of 52. What is the probability that you draw A 2 2 in that order?

Solve with conditional probability.

- \( A_1 = \) event 1st card is A
- \( A_2 = \) event 2nd card is 2
- \( A_3 = \) event 3rd card is 2

Want:

\[
P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2)
\]

\[
= \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{3}{50}
\]

**We can also solve this with the Counting Principle**

\# of cards: \( 52 \)  
\# A: \( 4 \)  
\# 2: \( 4 \)

\[
\]
Example 2 Radar Detection

An aircraft is present in a certain area with probability 0.05.

If an aircraft is present, the radar correctly detects it with probability 0.99.

If an aircraft is not present, the radar incorrectly registers it with probability 0.10.

What is the probability of a false alarm?

What is the probability of a missed detection?

False alarm would mean no aircraft but radar sees one.

Missed detection would be an aircraft and radar does not register it.

\( A = \) Air present \( B = \) Radar registers craft

Given \( P(A) = 0.05 \) \( P(B | A) = 0.99 \) \( P(B | A^c) = 0.10 \)

\( P(\text{false alarm}) = P(A^c \cap B) = P(A^c | B) P(B) = ??? \)

\[ = P(A^c \cap B) = P(B | A^c) P(A^c) \]

\[ \uparrow \text{ Symmetric} \Rightarrow \text{can flip} \]

\[ = (0.10)(0.95) = 0.095 \]

\[ = 0.095 \]
\[ P(B | A) = 0.99 \]
\[ P(B^c | A) = 0.01 \]

\[
P(\text{3 missed detections}) = P(A \land B^c) = P(B^c | A) P(A) = (0.01)(0.05) = 0.0005
\]

Ex: 12 undergrads and 4 grad students are divided randomly into 4 groups (with all possible divisions equally likely). What is the probability that each group has at least one grad student?

**Solution:**

\[ P(A_1 \land A_2 \land A_3 \land A_4) \]

\[ = P(A_1) P(A_2 | A_1) P(A_3 | A_1 \land A_2) P(A_4 | A_1 \land A_2 \land A_3) \]

\[ = \frac{12}{16} \cdot \frac{9}{15} \cdot \frac{12}{14} \cdot \frac{10}{13} \approx 0.140659341
\]
Solution 2

Condition Prob -> different set up

$\Pr(x_{i} = \exists \text{ group i has exactly one quad student})$

$\Pr(4 \text{ choose } 1 | A_{i}) = \frac{\text{Pr}(A_{i}) \cdot \text{Pr}(A_{2} | A_{1}) \cdot \text{Pr}(A_{3} | A_{2}) \cdot \text{Pr}(A_{4} | A_{3}) \cdot \text{Pr}(A_{5} | A_{4}) \cdot \text{Pr}(A_{6} | A_{5}) \cdot \text{Pr}(A_{7} | A_{6}) \cdot \text{Pr}(A_{8} | A_{7}) \cdot \text{Pr}(A_{9} | A_{8}) \cdot \text{Pr}(A_{10} | A_{9})}{\text{Pr}(\text{total})} = \frac{4 \cdot 12 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7} = \frac{1}{14060544}$

Solution 3 Counting

A = desired event

$\Pr(A) = \frac{\#A}{\#\Omega}$

$\#A = 12! \div (3!3!3!3!) \cdot 4!$

$\#\Omega = 10! \div (4!4!4!4!)$

$\Pr(A) = \frac{\#A}{\#\Omega} = \frac{12! \div (3!3!3!3!) \cdot 4!}{10! \div (4!4!4!4!)} = \frac{18! \cdot 4!}{3!3!3!3! \cdot 4!4!4!4!}$

$= \frac{14060544}{14060544} = 1$