most important tool that goes with cond. prob.

Bayes' Formula

**Version 1**

\[ P(A | B) = \frac{P(B | A) P(A)}{P(B)} \]

*Proof:* \( P(A | B) P(B) = P(A \cap B) = P(B | A) P(A) \)

Can't commute conditioning

**Version 2**

Break up B with Law of Total Probability

\[ P(A | B) = \frac{P(B | A) P(A)}{P(B)} \]

\[ = \frac{P(B | A) P(A)}{P(B | A) P(A) + P(B | A^c) P(A^c)} \]

*Proof:* Law of Total Prob says

\[ P(B) = P(B \cap A) + P(B \cap A^c) \]
Bayes' Formula

Version 1
\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

Proof: \[ P(A|B)P(B) = P(AB) = P(B|A)P(A) \]

Version 2
\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \]

Proof: Law of Total Probability Says
\[ P(B) = P(BA_1) + P(BA_2) = P(B|A_1)P(A_1) + P(B|A^c)P(A^c) \]

Version 3
Let \( A_1, A_2, \ldots \) be a countable partition of \( \Omega \)
\[ P(A_k|B) = \frac{P(B|A_k)P(A_k)}{P(B)} = \frac{P(B|A_k)P(A_k)}{\sum_i P(B|A_i)P(A_i)} \]

Proof: Law of Total Probability Says
\[ P(B) = \sum_i P(B|A_i) \Rightarrow P(B|A_i) = \frac{P(B|A_i)P(A_i)}{P(B)} \]
Example: Medical Test

Suppose 0.1% of the population carries a certain disease. For people with the disease, there is a test that correctly gives a positive result 99.8% of the time. For people without the disease, the test correctly gives a negative result 99.7% of the time.

If you test positive, what is the probability you actually have the disease?

\[ A = \text{have disease} \]
\[ B = \text{test positive} \]

\[ P(A \mid B) = P(\text{have disease} \mid \text{test positive}) \]

Given:

\[ P(A) = 0.001 \]
\[ P(A^c) = 0.999 \]

\[ P(B \mid A) = P(\text{test positive} \mid \text{have disease}) = 0.998 \]
\[ \text{[true positive rate = sensitivity]} \]

\[ P(B \mid A^c) = P(\text{test negative} \mid \text{have disease}) = 0.002 \]
\[ \text{[false negative rate]} \]

\[ P(B^c \mid A) = P(\text{test negative} \mid \text{no disease}) = 0.997 \]
\[ \text{[true negative rate = specificity]} \]

\[ P(B^c \mid A^c) = P(\text{test negative} \mid \text{no disease}) = 0.997 \]
$P(B|A^c) = P(\text{test positive } | \text{no disease}) = 0.003$

The desired probability

$P(A|B) = P(\text{have disease } | \text{test positive}) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$

$= \frac{0.998 (0.001)}{0.998 (0.001) + (0.003)(0.999)}$

$= 0.8498$

$= 84.98\%$

$P(B|A) = 1 - P(B^c | A)$

Bayes' Theorem:

$\frac{P(A \cap B)}{P(B)} = P(A|B)$

$\frac{P(A \cap B)}{P(A)} = P(B|A)$
Example:
Roll a fair 6-sided die
A = \{2, 4, 6\}
B = \{1, 2, 3, 4, 5\}

\[ P(B|A) = \frac{3}{5} \]
\[ P(B^c|A) = \frac{2}{5} = 1 - P(B|A) \]
\[ P(B|A^c) = \frac{2}{3} \neq 1 - P(B|A) \]

Bayes says you have to make changes to go to another world.

Example with Bayes: Grocery Store
The store gets eggs from 3 different farms
20% from farm 1,
30% from farm 2,
5% of the egg cartons from farm 1 contain a broken egg
3% from farm 2
2% from farm 3

What percentage contain a cracked egg?

If you open a carton you find a cracked egg, what is the prob. that carton came from farm 3?
\[ A_i = \sum \text{carton is from farm}_i \]
\[ B = \sum \text{carton has a cracked egg}_3 \]

Given
\[
P(A_1) = \frac{20}{100} \Rightarrow 0.2
\]
\[
P(A_2) = \frac{30}{100} \Rightarrow 0.3
\]
\[
P(A_3) = \frac{50}{100} \Rightarrow 0.5
\]
\[
P(B | A_1) = 0.05
\]
\[
P(B | A_2) = 0.03
\]
\[
P(B | A_3) = 0.02
\]

\[ A_1, A_2, A_3: \text{partition the sample space} \]
\[
P(B) = P(B | A_1) P(A_1) + P(B | A_2) P(A_2) + P(B | A_3) P(A_3)
\]
\[
= (0.05)(0.2) + (0.03)(0.3) + (0.02)(0.5)
\]
\[
= 0.024
\]

2.4\% of cartons have cracked eggs

\[
P(A_3 | B) = \frac{P(B | A_3) P(A_3)}{P(B)} = \frac{P(B | A_3) P(A_3)}{\sum_{i=1}^{3} P(B | A_i) P(A_i)}
\]
\[
= \frac{(0.02)(0.50)}{0.024}
\]
\[
= 0.344...
\]

Given a carton with a cracked egg:
34.4\% chance it's from farm 3