GLOBAL REGULARITY OF THE TWO-DIMENSIONAL MAGNETO-MICROPOLAR FLUID SYSTEM WITH ZERO ANGULAR VISCOSITY

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Abstract. We study the two-dimensional magneto-micropolar fluid system. Making use of the structure of the system, we show that with zero angular viscosity the solution triple remains smooth for all time.

Keywords: Magneto-micropolar fluid, micropolar fluid, global regularity, regularity criteria, Besov spaces

1. Introduction

The magneto-micropolar fluid (MMPF) system consists of the following equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u - r(b \cdot \nabla) b + \nabla (\pi + \frac{1}{2} r |b|^2) + (\mu + \chi) \Lambda^{2r_1} u = \chi (\nabla \times w), \quad (1a)$$

$$\rho \frac{\partial w}{\partial t} + \rho (u \cdot \nabla) w + \gamma \Lambda^{2r_2} w = -2\chi w + (\alpha + \beta) \nabla \text{div} w + \chi (\nabla \times u), \quad (1b)$$

$$\frac{\partial b}{\partial t} + (u \cdot \nabla) b - (b \cdot \nabla) u + \nu \Lambda^{2r_3} b = 0, \quad (1c)$$

$$\nabla \cdot u = \nabla \cdot b = 0, \quad (u, w, b) (x, 0) = (u_0, w_0, b_0) (x), \quad (1d)$$

where we denoted $u, w, b, \pi$ the velocity, micro-rotational velocity, the magnetic and the hydrostatic pressure fields respectively. We also denoted physically meaningful quantities: $r = \frac{M^2}{Re Rm}$ where $M$ is the Hartmann number, $Re$ the Reynolds number, $Rm$ the magnetic Reynolds number, $\chi$ the vortex viscosity, $\mu$ the kinematic viscosity, $\rho$ the microinertia, $\alpha, \beta, \gamma$ the angular viscosities, $\nu = \frac{1}{Rm}$ all of which we assume to be positive taking into account of conditions such as Clausius-Duhem inequality.

Finally, we denoted fractional Laplacians defined through Fourier transform by

$$\hat{\Lambda}^{r_i} f (\xi) = |\xi|^{r_i} \hat{f} (\xi), \quad i = 1, 2, 3.$$ 

Hereafter let us write $\partial_t = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial x_i}$ and also assume $\rho = 1$.

Let us discuss the rich history of mathematical and physical study concerning the MMPF system and its related systems. Firstly, the MMPF system at $w \equiv b \equiv 0$...
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reduces to the Navier-Stokes equations (NSE). The NSE has ample engineering applications in fluid mechanics and has been investigated mathematically with much intensity.

Secondly, the MMPF system at $\chi = 0, w \equiv 0$ reduces to the magnetohydrodynamics (MHD) system which describes the motion of electrically conducting fluids and has broad applications in applied sciences such as astrophysics, geophysics and plasma physics (cf. [23]). The mathematical analysis of the MHD system has attracted much attention in particular concerning the global regularity issue in two-dimensional case (cf. [3, 4, 17, 24, 27, 31, 32, 33, 38]).

Thirdly, the MMPF system at $b \equiv 0$ reduces to the micropolar fluid (MPF) system:

$$\partial_t u + (u \cdot \nabla) u + \nabla \pi + (\mu + \chi) \Lambda^2 u = \chi (\nabla \times w),$$
$$\partial_t w + (u \cdot \nabla) w + \gamma \Lambda^2 w = -2\chi w + (\alpha + \beta) \nabla \text{div} w + \chi (\nabla \times u),$$
$$\nabla \cdot u = 0, \quad (u, w)(x, 0) = (u_0, w_0)(x).$$

The microfluids and micropolar fluids were introduced in [13, 14] respectively. In particular, the micropolar fluids represent the fluids consisting of bar-like elements, e.g. anisotropic fluids, such as liquid crystals made up of dumbbell molecules and animal blood. The study of this system was continued by many (e.g. [5, 6, 8, 9, 10, 11, 18, 19, 29, 30, 37]). In particular, the authors in [12] obtained global regularity result of the two-dimensional MPF system with zero angular viscosity and $r_1 = 1$.

Finally, the MMPF system (1a)-(1d) was considered in [1] in which the authors obtained Serrin-type stability criteria. This system has also found much applications and attraction by mathematicians (cf. [16, 21, 22, 25, 28, 30, 35, 36, 39]). In particular, the authors in [15] obtained the global existence of weak solution triple $(u, w, b)$, global existence of strong solution triple $(u, w, b)$ in case initial data is small if dimension is three, while the unique weak solution triple $(u, w, b)$ if dimension is two, both with $r_1 = r_2 = r_3 = 1$.

We now consider, following the work of [20, 12], the two-dimensional problem by setting

$$u = (u_1, u_2, 0), \quad w = (0, 0, w_3)$$

and denote by

$$\Omega := \nabla \times u = \partial_1 u_2 - \partial_2 u_1, \quad j := \nabla \times b = \partial_1 b_2 - \partial_2 b_1, \quad \nabla \times w = (\partial_2 w, -\partial_1 w).$$

We present our results concerning the following special case of (1) with $x \in \mathbb{R}^2$:

$$\partial_t u + (u \cdot \nabla) u - (b \cdot \nabla) b + \nabla (\pi + \frac{1}{2} |b|^2) = (\mu + \chi) \Delta u + \chi (\nabla \times w),$$
$$\partial_t w + (u \cdot \nabla) w = -2\chi w + \chi (\nabla \times u),$$
$$\partial_t b + (u \cdot \nabla) b - (b \cdot \nabla) u = \nu \Delta b.$$

**Theorem 1.1.** For every $(u_0, w_0, b_0) \in H^s(\mathbb{R}^2), s > 2$, there exists a unique solution triple $(u, w, b)$ to (3a)-(3c) such that

$$u, b \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+1}(\mathbb{R}^2)), \quad w \in C([0, \infty); H^s(\mathbb{R}^2)).$$
Remark 1.1. 

(1) Theorem 1.1 improves previous global regularity results with angular viscosity (cf. [20]) and extend those in [12] from MPF system to the MMPF system. Furthermore, it is also an extension of the global well-posedness result of the two-dimensional classical MHD system (cf. [23]).

(2) In three-dimension, employing standard energy method on (1a)-(1d), one can show that the global regularity is attained as long as

\[ r_1 \geq \frac{2 + N}{4}, \quad \min\{r_1, r_2, r_1 + r_3\} \geq \frac{2 + N}{2}, \quad N = 3, \]

(cf. [26]). Hence, the result of Theorem 1.1 at \( r_1 = r_3 = 1, r_2 = 0 \) is a significant improvement, attainable due to the advantage of the structure of the system upon taking curls, very similarly to the recent developments in the two-dimensional generalized MHD system.

(3) Local existence proof can be performed in many ways, for example using mollifiers following the work on the NSE. Because the authors in [12] proved the local existence in the case of the MPF system in detail and one can follow their work line by line, just adding the magnetic field equation, we omit this proof.

Let us use the notation \( A \lesssim_{a,b} B, A \approx_{a,b} B \) to imply that there exists a non-negative constant \( c \) that depends on \( a, b \) such that \( A \leq cB, A = cB \) and briefly discuss the difficulty of this problem before we present our proof.

Firstly, we take \( L^2 \)-inner products with \((u,w,b)\) on (3a)-(3c) respectively to obtain due to the incompressibility of \( u \),

\[
\frac{1}{2} \partial_t \|u\|_{L^2}^2 + (\mu + \chi) \|\nabla u\|_{L^2}^2 = \int (b \cdot \nabla) b \cdot u + \chi \int (\nabla \times w) \cdot u \\
\frac{1}{2} \partial_t \|w\|_{L^2}^2 + 2\chi \|w\|_{L^2}^2 = \chi \int (\nabla \times u) w \\
\frac{1}{2} \partial_t \|b\|_{L^2}^2 + \nu \|
abla b\|_{L^2}^2 = \int (b \cdot \nabla) u \cdot b
\]

In sum, using the fact that \( \int (\nabla \times w) \cdot u = \int (\nabla \times u) w \), we obtain

\[
\frac{1}{2} \partial_t (\|u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|b\|_{L^2}^2) + (\mu + \chi) \|\nabla u\|_{L^2}^2 + 2\chi \|w\|_{L^2}^2 + \nu \|\nabla b\|_{L^2}^2 \\
\leq 2\chi \|w\|_{L^2} \|\nabla u\|_{L^2} \leq 2\chi \|w\|_{L^2}^2 + \frac{K}{2} \|\nabla u\|_{L^2}^2
\]

by the incompressibility of \( b \), Hölder’s and Young’s inequalities. Therefore, after absorbing the right hand side, integrating in time we obtain

\[
\sup_{t \in [0,T]} \left( \|u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|b\|_{L^2}^2 \right) (t) + \int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 d\tau \lesssim_{u_0, w_0, b_0} 1. \quad (4)
\]

We will obtain via a commutator estimate a regularity criteria that in order to prove Theorem 1.1, it suffices to show the time integrability of \( \|\Omega\|_{L^\infty}, \|w\|_{L^2}, \|\nabla j\|_{L^2} \) (See Proposition 3.2). Now taking a curl on (3a) leads to

\[
\partial_t \Omega - (\mu + \chi) \Delta \Omega = -(u \cdot \nabla) \Omega + (b \cdot \nabla) j - \chi \Delta w. \quad (5)
\]
Due to the lack of angular viscosity, we have no obvious way to handle $-\chi \Delta w$. The key observation in [12] was that by defining

$$Z := \Omega - \left( \frac{\chi}{\mu + \chi} \right) w,$$

we can take advantage of its evolution in time governed by

$$\partial_t Z = (\mu + \chi) \Delta Z - (u \cdot \nabla) Z - c_1 Z + c_2 w + (b \cdot \nabla) j$$

where we denoted

$$c_1 := \frac{\chi^2}{\mu + \chi} \geq 0, \quad c_2 := \frac{2\chi^2}{\mu + \chi} - \frac{\chi^3}{(\mu + \chi)^2}.$$

This leads to the following $L^p$-estimate of $Z$ after multiplying (7) by $|Z|^{p-2} Z$, integrating in space, using the incompressibility of $u$ and Hölder’s inequality:

$$\frac{1}{p} \partial_t ||Z||_{L^p}^p \lesssim ||w||_{L^p} ||Z||_{L^p}^{p-1} + ||(b \cdot \nabla) j||_{L^p} ||Z||_{L^p}^{p-1}.$$  

Dividing by $||Z||_{L^p}^{p-1}$, we obtain taking limit $p \to \infty$,

$$\partial_t ||Z||_{L^\infty} \lesssim ||w||_{L^\infty} + ||(b \cdot \nabla) j||_{L^\infty} \lesssim ||w||_{L^\infty} + ||b||_{L^\infty} ||\nabla j||_{L^\infty}. $$

On the other hand, from (3b) we can estimate similarly

$$\frac{1}{p} \partial_t ||w||_{L^p}^p + 2\chi ||w||_{L^p}^p \leq \chi ||\Omega||_{L^p} ||w||_{L^p}^{p-1} \lesssim ||Z||_{L^p} ||w||_{L^p}^{p-1} + ||w||_{L^p}^p,$$

by the incompressibility of $u$, Hölder’s inequality and (6). Dividing by $||w||_{L^p}^{p-1}$ and taking limit $p \to \infty$ lead to

$$\partial_t ||w||_{L^\infty} \lesssim ||Z||_{L^\infty} + ||w||_{L^\infty}.$$  

In sum of (8) and (9) we obtain

$$\partial_t (||Z||_{L^\infty} + ||w||_{L^\infty}) \lesssim ||Z||_{L^\infty} + ||w||_{L^\infty} + ||b||_{L^\infty} ||\nabla j||_{L^\infty}.$$ 

Thus, in the absence of $(b \cdot \nabla) j$, the authors in [12] were able to show the global regularity for the two-dimensional MPF system with zero angular viscosity. If we replace the diffusive term $-\Delta b$ by $A^{2\alpha} b$ so that this becomes familiar to the recent developments in the two-dimensional generalized MHD system (cf. [31]), then this problem in our case requires, according to (4), $r_3 > 3$ considering a Sobolev embedding $H^{r_3-2}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. A natural idea to reduce the lower bound on $r_3$ is to improve the bound on $b$. To create some cancellations, it is optimal to estimate $\Omega$ and $j$ together. Because taking a curl on (3c) gives

$$\partial_t j - \nu \Delta j = -(u \cdot \nabla) j + (b \cdot \nabla) \Omega + 2[\partial_t b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)],$$

we have the $L^2$-estimates of $\Omega$ and $j$ as follows:

$$\frac{1}{2} \partial_t ||\Omega||_{L^2}^2 + (\mu + \chi) ||\nabla \Omega||_{L^2}^2 = \int (b \cdot \nabla) j \Omega - \chi \Delta w \Omega,$$

$$\frac{1}{2} \partial_t ||j||_{L^2}^2 + \nu ||\nabla j||_{L^2}^2 = \int (b \cdot \nabla) \Omega j + 2[\partial_t b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)] j.$$
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Considering the term $\chi \int \Delta w \Omega$, we have no choice but to try the estimate on $w$ simultaneously. However, taking $L^2$-inner products of (3b) with $-\Delta w$ leads to

$$\frac{1}{2} \partial_t \| \nabla w \|^2_{L^2} + 2 \chi \| \nabla w \|^2_{L^2} = - \int \nabla u \cdot \nabla w \cdot \nabla w + \chi \int \nabla \Omega \cdot \nabla w,$$

after integration by parts and the first nonlinear term becomes problematic. The novelty of this manuscript is the observation that by defining $Z$ in (6), a certain bound of $j$ will lead to a bound of $Z$, which in turn leads to the bound of $w$, and then $\Omega$.

In the next section, we set up notations and state key lemmas. Thereafter we prove our result. The proof consists of two parts: a regularity criteria for the solution triple to (3a)-(3c) using a commutator estimate and an a priori estimate making use of $Z$ in (6).

2. PRELIMINARIES

We state some key lemmas and facts that will be crucial in our proofs.

Lemma 2.1. (cf. [7]) Let $f$ be divergence-free vector field such that $\nabla f \in L^p$, $p \in (1, \infty)$. Then

$$\| \nabla f \|_{L^p} \leq c \frac{p^2}{p-1} \| \text{curl } f \|_{L^p}.$$ 

The next lemma consists of variations of the classical result from [2]; for the proofs of these specific inequalities we refer readers to the Appendices of [31, 34]:

Lemma 2.2. Suppose $f \in L^2(\mathbb{R}^2) \cap H^s(\mathbb{R}^2), s > 2$. Then

$$\| \nabla f \|_{L^\infty} \leq c (\| f \|_{L^2} + \| \text{curl } f \|_{L^\infty} \log_2 (2 + \| f \|_{H^s}) + 1), \quad (11)$$

$$\| \nabla f \|_{L^\infty} \leq c (\| f \|_{L^2} + \| \nabla f \|_{H^1} \log_2 (2 + \| f \|_{H^s}) + 1). \quad (12)$$

Let us recall the notion of Besov spaces (cf. [7]). We denote by $\mathcal{S}(\mathbb{R}^2)$ the Schwartz class functions and $\mathcal{S}'(\mathbb{R}^2)$, its dual. We define $\mathcal{S}_0$ to be the subspace of $\mathcal{S}$ in the following sense:

$$\mathcal{S}_0 = \{ \phi \in \mathcal{S}, \int_{\mathbb{R}^2} \phi(x)x^l dx = 0, |l| = 0, 1, 2, ... \}.$$ 

Its dual $\mathcal{S}'_0$ is given by $\mathcal{S}'_0 = \mathcal{S}/\mathcal{S}_0 \dagger = \mathcal{S}'/\mathcal{P}$ where $\mathcal{P}$ is the space of polynomials. For $k \in \mathbb{Z}$ we define

$$A_k = \{ \xi \in \mathbb{R}^2 : 2^{k-1} < |\xi| < 2^{k+1} \}.$$ 

It is well-known that there exists a sequence $\{ \Phi_k \} \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\text{supp } \hat{\Phi}_k \subset A_k, \quad \hat{\Phi}_k(\xi) = \Phi_0(2^{-k} \xi) \quad \text{or} \quad \Phi_k(x) = 2^{2k} \Phi_0(2^k x)$$

and

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}_k(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^2 \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Consequently, for any $f \in \mathcal{S}'_0$,
\[
\sum_{k=\infty}^{\infty} \Phi_k * f = f.
\]

To define the homogeneous Besov spaces, we set
\[
\hat{\Delta}_k f = \Phi_k * f, \quad k = 0, \pm 1, \pm 2, \ldots.
\]
With such we can define for \( s \in \mathbb{R}, p, q \in [1, \infty] \), the homogeneous Besov spaces
\[
\dot{B}^s_{p,q} = \{ f \in S'_0 : \| f \|_{\dot{B}^s_{p,q}} < \infty \},
\]
where \( \| f \|_{\dot{B}^s_{p,q}} = \left( \sum_{k} (2^{ks} \| \hat{\Delta}_k f \|_{L^p})^q \right)^{\frac{1}{q}} \) if \( q < \infty \),
\[
\sup_{k \geq 1} 2^{ks} \| \hat{\Delta}_k f \|_{L^p} \quad \text{if } q = \infty.
\]
To define the inhomogeneous Besov spaces, we let \( \Psi \in C_0^\infty(\mathbb{R}^2) \) be such that
\[
1 = \hat{\Psi}(\xi) + \sum_{k=0}^{\infty} \hat{\Phi}_k(\xi), \quad \Psi * f + \sum_{k=0}^{\infty} \Phi_k * f = f,
\]
for any \( f \in S' \). With that, we set
\[
\Delta_k f = \begin{cases} 
0 & \text{if } k \leq -2,
\Psi * f & \text{if } k = -1,
\Phi_k * f & \text{if } k = 0, 1, 2, \ldots,
\end{cases}
\]
and define for any \( s \in \mathbb{R}, p, q \in [1, \infty] \), the inhomogeneous Besov spaces
\[
B^s_{p,q} = \{ f \in S' : \| f \|_{B^s_{p,q}} < \infty \},
\]
where \( \| f \|_{B^s_{p,q}} = \left( \sum_{k = -1}^{\infty} \left( 2^{ks} \| \Delta_k f \|_{L^p} \right)^q \right)^{\frac{1}{q}} \) if \( q < \infty \),
\[
\sup_{-1 \leq k < \infty} 2^{ks} \| \Delta_k f \|_{L^p} \quad \text{if } q = \infty.
\]
For any \( s \in \mathbb{R}, 1 < p < \infty \), we have
\[
B^s_{p,\min\{p,2\}} \subset W^s,p \subset B^s_{p,\max\{p,2\}}, \quad \dot{B}^s_{p,\min\{p,2\}} \subset \dot{W}^s,p \subset \dot{B}^s_{p,\max\{p,2\}}.
\]
Moreover, Bony’s paraproduct decomposition will be used frequently (cf. [7])
\[
f g = T_f g + T_g f + R(f, g) \quad \text{where}
\]
\[
T_f g = \sum_k S_{k-1} \Delta_k g, \quad R(f, g) = \sum_{k, k': |k-k'| \leq 1} \Delta_k f \Delta_{k'} g, \quad S_{k-1} = \sum_{m: m \leq k - 2} \Delta_m.
\]

3. Commutator estimate and regularity criteria

In this section, we obtain a regularity criteria for the solution \((u, w, b)\) to (3a)-(3c) using a commutator estimate, namely the following:
Proposition 3.1. Let $s > -1$, $f \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\nabla g \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\nabla \cdot g = 0$. Then for any $k \geq 3$,
\[
\|\Delta_k, g^i \partial_i f\|_{L^2} \lesssim c_k 2^{-ks}(\|\nabla g\|_{L^\infty} \|f\|_{H^s} + \|\nabla g\|_{H^s} \|f\|_{L^\infty}). \tag{13}
\]
Moreover, for $s > 0$ if additionally $\nabla f \in L^\infty(\mathbb{R}^2)$, then for any $k \geq -1$,
\[
\|\Delta_k, g^i \partial_i f\|_{L^2} \lesssim c_k 2^{-ks}(\|\nabla g\|_{L^\infty} \|f\|_{H^s} + \|g\|_{H^s} \|\nabla f\|_{L^\infty}) \tag{14}
\]
where \(\{c_k\} \in l^2_{k \geq -1}\).

Proof. The inequality (13) is shown in [12]; we focus on (14). We first write
\[
[\Delta_k, g^i \partial_i] f = \Delta_k \left( \sum_{k' : |k-k'| \leq 2} S_{k' - 1} g^i \Delta_{k'} \partial_i f \right) - \sum_{k' : |k-k'| \leq 2} (S_{k' - 1} g^i) \Delta_{k'}(\Delta_k \partial_i f).
\]
We rewrite this as
\[
\Delta_k \left( \sum_{k' : |k-k'| \leq 2} S_{k' - 1} g^i \Delta_{k'} \partial_i f \right) - \sum_{k' : |k-k'| \leq 2} (S_{k' - 1} g^i) \Delta_{k'}(\Delta_k \partial_i f)
\]
\[
= \sum_{k' : |k-k'| \leq 2} [((S_{k' - 1} g^i) \Delta_{k'}(\Delta_k \partial_i f) - \Delta_k(S_{k' - 1} g^i \Delta_{k'} \partial_i f)]
\]
\[
= \sum_{k' : |k-k'| \leq 2} 2^{2k} \int_{\mathbb{R}^2} \Phi_0(2^k y) \left[ \int_0^1 \frac{\partial}{\partial \tau} S_{k' - 1} g^i(x - \tau y) d\tau \right] \frac{\partial}{\partial y_i} \Delta_{k'} f(x - y) dy.
\]
by definition of $\Delta_k$ and integration by parts using divergence-free property of $g$. Thus,
\[
\|([\Delta_k, g^i \partial_i] f)\|_{L^2} \lesssim \sum_{k' : |k-k'| \leq 2} \|\nabla S_{k' - 1} g^i\|_{L^\infty} \left\| 2^{3k} x \frac{\partial \Phi_0(2^k x)}{\partial (2^k x)} \right\|_{L^1} \|\Delta_{k'} f\|_{L^2}
\]
by Young’s inequality for convolution. Hence, because $\Phi_0 \in \mathcal{S}(\mathbb{R}^2),$
\[
\|([\Delta_k, g^i \partial_i] f)\|_{L^2} \lesssim \|\nabla g\|_{L^\infty} \|\nabla f\|_{L^2} \lesssim c_k 2^{-ks} \|\nabla g\|_{L^\infty} \|f\|_{H^s}, \tag{16}
\]
where we used that under restriction of $|k-k'| \leq 2$, we may replace $k'$ by $k$ modifying constants. Next,
by Hölder’s inequality. Next,

\begin{align*}
\|\Delta_k R(g^i, \partial_i f)\|_{L^2} &\lesssim \sum_{k', k \leq k} \|\Delta_k g^i \Delta_k \partial_i f\|_{L^2} \\
&\lesssim \|\nabla f\|_{L^\infty} 2^{-ks} 2^{k'} \|\Delta_k g\|_{L^2} \lesssim c_k 2^{-ks} \|\nabla f\|_{L^\infty} \|g\|_{H^*}
\end{align*}

by Hölder’s inequality. Next,

\begin{align*}
\|T \partial_k g\|_{L^2} = &\sum_{k', k \leq k'} \sum_{k, k' \leq k'} (\Delta_k \partial_k g) \Delta_k g^i \|L^2\| \\
&\lesssim \sum_{k', k \leq k'} \|\Delta_k \partial_k f \Delta_k g^i\|_{L^2} \lesssim c_k 2^{-ks} \|\nabla f\|_{L^\infty} \|g\|_{H^*}
\end{align*}

where we used that \(\Delta_k \partial_k f = 0\) if \(|k' - k| > 2\) and Hölder’s inequality. Finally,

\begin{align*}
\|R(g^i, \partial_i \Delta_k f)\|_{L^2} &\lesssim \sum_{k', k'} \|\Delta_k g^i \Delta_k \partial_i \Delta_k f\|_{L^2} \\
&\lesssim \|\nabla f\|_{L^\infty} \|\Delta_k g\|_{L^2} \lesssim c_k 2^{-ks} \|\nabla f\|_{L^\infty} \|g\|_{H^*}
\end{align*}

by Hölder’s inequalities. Taking \(L^2\)-norm on (15) and considering (16)-(20), we obtain

\[\|\Delta_k g^i \partial_i f\|_{L^2} \lesssim c_k 2^{-ks} (\|\nabla f\|_{L^\infty} \|g\|_{H^*} + \|\nabla g\|_{L^\infty} \|f\|_{H^*}).\]

This completes the proof of Proposition 3.1.

\[\square\]

**Proposition 3.2.** Suppose \((u_0, w_0, b_0) \in H^s(\mathbb{R}^2), s > 2\) and its corresponding solution triple \((u, w, b)\) to (3a)-(3c) in time interval \([0, T]\) satisfies

\[\int_0^T \|\Omega\|_{L^\infty} + \|w\|_{L^\infty}^2 + \|\nabla j\|_{L^2}^2 \, dt \lesssim 1\]

where \(\Omega = \nabla \times u, j = \nabla \times b\). Then \((u, w, b) \in H^s(\mathbb{R}^2)\) for all time \(t \in [0, T]\).

**Proof.** We apply \(\Delta_k, k \geq -1\) and take \(L^2\)-inner products of (3a)-(3c) with \((\Delta_k u, \Delta_k w, \Delta_k b)\) respectively and sum to obtain
\[
\frac{1}{2} \partial_t (\|\Delta_k u\|_{L^2}^2 + \|\Delta_k w\|_{L^2}^2 + \|\Delta_k b\|_{L^2}^2 ) \\
+ (\mu + \chi) \|\nabla \Delta_k u\|_{L^2}^2 + 2\chi \|\Delta_k w\|_{L^2}^2 + \nu \|\nabla \Delta_k b\|_{L^2}^2 \\
= - \int \Delta_k ((u \cdot \nabla) u) \cdot \Delta_k u - (u \cdot \nabla) \Delta_k \cdot \Delta_k u \\
+ \int \Delta_k ((b \cdot \nabla) b) \cdot \Delta_k u - (b \cdot \nabla) \Delta_k b \cdot \Delta_k u \\
- \int \Delta_k ((u \cdot \nabla) w) \cdot \Delta_k w - (u \cdot \nabla) \Delta_k w \cdot \Delta_k w \\
- \int \Delta_k ((u \cdot \nabla) b) \cdot \Delta_k b - (u \cdot \nabla) \Delta_k b \cdot \Delta_k \\
+ \int \Delta_k ((b \cdot \nabla) u) \cdot \Delta_k b - (b \cdot \nabla) \Delta_k u \cdot \Delta_k b + 2\chi \int \Delta_k w \cdot \Delta_k (\nabla \times u)
\]

where we used the incompressibility of \(u\) and \(b\). For \(k \geq 3\), we now bound

\[
\frac{1}{2} \partial_t (\|\Delta_k u\|_{L^2}^2 + \|\Delta_k w\|_{L^2}^2 + \|\Delta_k b\|_{L^2}^2 ) \\
+ (\mu + \chi) \|\nabla \Delta_k u\|_{L^2}^2 + 2\chi \|\Delta_k w\|_{L^2}^2 + \nu \|\nabla \Delta_k b\|_{L^2}^2 \\
\leq \|\Delta_k, u \cdot \nabla) u\|_{L^2} \|\Delta_k u\|_{L^2} + \|\Delta_k, b \cdot \nabla) b\|_{L^2} \|\Delta_k u\|_{L^2} \\
+ \|\Delta_k, u \cdot \nabla) w\|_{L^2} \|\Delta_k w\|_{L^2} + \|\Delta_k, b \cdot \nabla) b\|_{L^2} \|\Delta_k b\|_{L^2} \\
+ \|\Delta_k, b \cdot \nabla) u\|_{L^2} \|\Delta_k u\|_{L^2} + 2\chi \|\Delta_k w\|_{L^2} \|\nabla \Delta_k u\|_{L^2} \\
\leq c_k 2^{-k\delta} (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} + \|\Delta_k u\|_{L^2} + \|\Delta_k b\|_{L^2}) \|\Delta_k u\|_{L^2} \\
+ (\|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} + \|\Delta_k b\|_{L^2}) \|\Delta_k u\|_{L^2} \\
+ \|\Delta_k u\|_{L^\infty} \|\Delta_k b\|_{L^2} + \|\nabla b\|_{L^\infty} \|\Delta_k b\|_{L^2} ) + 2\chi \|\Delta_k w\|_{L^2}^2 + \frac{\chi}{2} \|\nabla \Delta_k u\|_{L^2}^2
\]

by Hölder’s inequalities, (14) on the first, second, fourth and fifth commutators and (13) on the third and Young’s inequalities. Absorbing the last two terms, multiplying by \(2^{ks}\), \(k \geq 3\), \(s \geq 2\), we obtain

\[
\partial_t 2^{ks} (\|\Delta_k u\|_{L^2}^2 + \|\Delta_k w\|_{L^2}^2 + \|\Delta_k b\|_{L^2}^2 ) \\
+ (\mu + \chi) 2^{ks} \|\nabla \Delta_k u\|_{L^2}^2 + 2\nu 2^{2ks} \|\nabla \Delta_k b\|_{L^2}^2 \\
\leq c_k 2^{ks} (\|\Delta_k u\|_{L^2} + \|\Delta_k w\|_{L^2} + \|\Delta_k b\|_{L^2}) \\
	imes (\|u\|_{H^s} + \|w\|_{H^s} + \|b\|_{H^s} + 1) (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} + 1) \\
+ c_k 2^{ks} \|\nabla u\|_{H^s} \|w\|_{L^\infty} \|\Delta_k w\|_{L^2}.
\]
For the case $k = -1, 0, 1, 2$, we use the fact that $\Delta_k$ is a convolution operator, $\Psi, \Phi_k \in \mathcal{S}(\mathbb{R}^2)$, Hölder’s and Young’s inequalities for convolution and (4) to obtain

$$\sum_{k=-1}^{2} 2^{2ks} \int -\Delta_k((u \cdot \nabla)u) \cdot \Delta_k u + \int \Delta_k((b \cdot \nabla)b) \cdot \Delta_k u - \int \Delta_k((u \cdot \nabla)w) \cdot \Delta_k w$$

$$- \int \Delta_k((u \cdot \nabla)b) \cdot \Delta_k b + \int \Delta_k((b \cdot \nabla)u) \cdot \Delta_k b + 2\chi \int \Delta_k w \cdot \Delta_k (\nabla \times u)$$

$$\lesssim \sum_{k=-1}^{2} 2^{2ks}(\|u \otimes u\|_{L^1} \|u\|_{L^2} + \|b \otimes b\|_{L^1} \|u\|_{L^2} + \|u \otimes u\|_{L^1} \|w\|_{L^2} + \|b \otimes u\|_{L^1} \|b\|_{L^2} + \|w\|_{L^2} \|u\|_{L^2} ) \lesssim 1.$$ 

Thus, we can multiply (21) by $2^{2ks}$, sum over $k \geq -1$ and apply Hölder’s inequalities to obtain

$$\partial_t(\|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|b\|_{H^s}^2) + (\mu + \chi)\|\nabla u\|_{H^s}^2 + 2\nu\|\nabla b\|_{H^s}^2$$

$$\lesssim (1 + \|c_k\|_\infty)(\|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|b\|_{H^s}^2 + 1)(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} + 1)$$

$$+ (1 + \|c_k\|_\infty)\|\nabla u\|_{H^s} \|w\|_{L^\infty} \|w\|_{H^s}.$$ 

For the last term on the right hand side, we take Young’s inequalities to obtain

$$c(1 + \|c_k\|_\infty)\|\nabla u\|_{H^s} \|w\|_{L^\infty} \|w\|_{H^s} \leq \frac{\mu + \chi}{2} \|\nabla u\|_{H^s}^2 + c\|w\|_{L^\infty}^2 \|w\|_{H^s}^2.$$ 

We apply (23) to (22), absorb the first term on the right hand side of (23) to obtain

$$\partial_t(\|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|b\|_{H^s}^2)$$

$$\lesssim (\|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|b\|_{H^s}^2 + 1)(\|\nabla u\|_{L^\infty} + \|w\|_{L^\infty} + \|\nabla b\|_{L^\infty} + 1).$$ 

By (11) of Lemma 2.2 applied to $\|\nabla u\|_{L^\infty}$, (12) of Lemma 2.2 applied to $\|\nabla b\|_{L^\infty}$ and (4) we obtain

$$\partial_t \log_2(2 + \|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|b\|_{H^s}^2)$$

$$\lesssim \log_2(2 + \|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|b\|_{H^s}^2)(1 + \|\Omega\|_{L^\infty} + \|w\|_{L^\infty} + \|\nabla j\|_{L^2}).$$ 

Gronwall’s inequality completes the proof of Proposition 3.2. 

\[ \square \]

4. A priori estimate

We consider $Z$ from (6) and first obtain $L^2$-estimates.

**Proposition 4.1.** Suppose $(u, w, b)$ solves (3a)-(3c) in time interval $[0, T]$. Then

$$\sup_{t \in [0, T]} (\|Z\|_{L^2}^2 + \|\Omega\|_{L^2}^2 + \|j\|_{L^2}^2)(t) + \int_0^T \|\nabla Z\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 d\tau \lesssim 1.$$
Proof. We take $L^2$-inner products (7) and (10) with $(Z,j)$ respectively to obtain
\[
\frac{1}{2} \partial_t \|Z\|_{L^2}^2 + (\mu + \chi) \|\nabla Z\|_{L^2}^2 + c_1 \|Z\|_{L^2}^2 \\
\leq |c_2| \|w\|_{L^2} \|Z\|_{L^2} + \int (b \cdot \nabla) j \left( \Omega - \left( \frac{\chi}{\mu + \chi} \right) w \right),
\]
(24)
\[
\frac{1}{2} \partial_t \|j\|_{L^2}^2 + \nu \|\nabla j\|_{L^2}^2 \\
= \int (b \cdot \nabla) \Omega j + 2 [\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)] j.
\]
(25)
where we used (6). We sum (24) and (25) and bound by H"older's and Gagliardo-Nirenberg inequalities, (4), Lemma 2.1, Young's inequalities and (6). By Gronwall's inequality and (4), this implies
\[
\frac{1}{2} \partial_t (\|Z\|_{L^2}^2 + \|j\|_{L^2}^2) + (\mu + \chi) \|\nabla Z\|_{L^2}^2 + \nu \|\nabla j\|_{L^2}^2 + c_1 \|Z\|_{L^2}^2 \\
\lesssim \|Z\|_{L^2}^2 + \|b\|_{L^2} \|\nabla j\|_{L^2}^2 \|w\|_{L^2} + \|\nabla b\|_{L^2} \|\nabla w\|_{L^2} \|j\|_{L^2} \\
\lesssim \|Z\|_{L^2}^2 + \|b\|_{L^2} \|\nabla j\|_{L^2}^2 + \|j\|_{L^2} \|\nabla j\|_{L^2} \|\nabla w\|_{L^2} \\
\lesssim \nu \|\nabla j\|_{L^2}^2 + c(\|Z\|_{L^2}^2 + \|j\|_{L^2}^2) \\
= \nu \|\nabla j\|_{L^2}^2 + c(\|Z\|_{L^2}^2 + \|j\|_{L^2}^2) (1 + \|j\|_{L^2}^2)
\]
by H"older's and Gagliardo-Nirenberg inequalities, (4), Lemma 2.1, Young's inequalities and (6). By Gronwall's inequality and (4), this implies
\[
\sup_{t \in [0,T]} (\|Z\|_{L^2}^2 + \|j\|_{L^2}^2) (t) + \int_0^T \|\nabla Z\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 \, dt \lesssim 1.
\]
Because we have the bound on $\|w(t)\|_{L^2}^2$ from (4), by (6) we obtain the bound on $\|\Omega(t)\|_{L^2}^2$. Thus, the proof of Proposition 4.1 is complete.
\[
□
\]
We obtain slightly higher integrability now:

**Proposition 4.2.** Suppose $(u,w,b)$ solves (3a)-(3c) in time interval $[0,T]$. Then for any $2 \leq q \leq 4$,
\[
\sup_{t \in [0,T]} (\|Z\|_{L^q} + \|\Omega\|_{L^q} + \|j\|_{L^q}) (t) \lesssim 1.
\]
(26)

Proof. We assume $2 < q \leq 4$ below as the case $q = 2$ is already done in the Proposition 4.1. We multiply (7) by $|Z|^{q-2} Z$, integrate in space to obtain
\[
\frac{1}{q} \partial_t \|Z\|_{L^q}^q - (\mu + \chi) \int \Delta Z |Z|^{q-2} Z + c_1 \|Z\|_{L^q}^q \\
= c_2 \int w |Z|^{q-2} Z + \int (b \cdot \nabla) j |Z|^{q-2} Z.
\]
(27)
For the dissipative term, we integrate by parts to obtain
\[-(\mu + \chi) \int \Delta Z |Z|^q Z = \frac{4(\mu + \chi)(q - 1)}{q^2} \| \nabla |Z|^\frac{q}{2} \|_{L^2}^2. \]  

(28)

On the other hand,

\[ \| Z \|_{L^2(q-1)} = \| |Z|^\frac{q}{2} \|_{L^2} \| \nabla |Z|^\frac{q}{2} \|_{L^2}^2 \lesssim \| \nabla |Z|^\frac{q}{2} \|_{L^2}^2 \| \nabla |Z|^\frac{q}{2} \|_{L^2}^2 \lesssim \| \nabla |Z|^\frac{q}{2} \|_{L^2}^2 \]  

where we used the Gagliardo-Nirenberg inequality and Proposition 4.1. This implies

\[ \| Z \|_{L^2(q-1)}^{-1} \lesssim \| \nabla |Z|^\frac{q}{2} \|_{L^2}^2. \]  

(29)

Now

\[ c_2 \int |w||Z|^q Z + \int (b \cdot \nabla) j |Z|^q Z \leq c \| w \|_{L^2} \| Z \|_{L^2(q-1)}^{-1} - (q - 1) \int b \cdot (\nabla Z) |Z|^q j \]  

(30)

by Hölder's inequality, (4), (29), integration by parts and Young's inequality. We can absorb the dissipative term whereas for the second term we estimate

\[ \int |b||Z|^q j^2 \lesssim \| b \|_{L^\infty} \| Z \|_{L^2} \| j \|_{L^2}^2 \]  

(31)

\[ \lesssim \| b \|_{L^2} \| \nabla j \|_{L^2} \| Z \|_{L^2}^{-1} \| j \|_{L^2}^{2(q-1)} \lesssim \| \nabla j \|_{L^2}^{1+2(q-1)} (\| Z \|_{L^2} + 1) \]

by Hölder's and Gagliardo-Nirenberg inequalities, (4) and Proposition 4.1. Thus, we have after absorbing, from (27), (28), (30) and (31),

\[ \frac{1}{q} \partial_t \| Z \|_{L^q}^q + \frac{2(\mu + \chi)(q - 1)}{q^2} \| \nabla |Z|^\frac{q}{2} \|_{L^2}^2 \lesssim (\| \nabla j \|_{L^2}^2 + 1) (\| Z \|_{L^q}^q + 1) \]  

(32)

by Young's inequality. Thus, by Proposition 4.1 and Gronwall's inequality applied to (32) we obtain

\[ \sup_{t \in [0,T]} \| Z(t) \|_{L^q} \lesssim 1. \]  

(33)

Next, we multiply (3b) by \(|w|^q - w\), integrate in space to obtain

\[ \frac{1}{q} \partial_t \| w \|_{L^q}^q + 2\chi \| w \|_{L^q}^q \lesssim \Omega \| w \|_{L^q}^{q-1} \lesssim \| Z \|_{L^q} \| w \|_{L^q}^{q-1} + \| w \|_{L^q}^{q-1} \lesssim (1 + \| w \|_{L^q}^q) \]

by Hölder's inequalities, (6) and (33). This implies by Gronwall's inequality

\[ \sup_{t \in [0,T]} \| w(t) \|_{L^q} \lesssim 1 \]

and hence by (6) and (33),
Finally, we multiply (10) by $|j|^{q-2}j$ and integrate in space to estimate

$$\frac{1}{q} \partial_t \|j\|_{L^q}^q - \nu \int \Delta j |j|^{q-2} j$$

and estimate the first non-linear term of (35) by integration by parts and Young’s inequality as follows:

$$-\nu \int \Delta j |j|^{q-2} j = \nu(q - 1) \int \nabla j \cdot \nabla |j|^{q-2} = \nu(q - 1) \int \left| \nabla |j|^{\frac{q-2}{2}} \right|^2$$

On the diffusive term, we apply integration by parts,

$$|b| \int \frac{q}{2} \partial_t \|j\|_{L^q}^q = -\nu \int \Delta j |j|^{q-2} j$$

and estimate the first non-linear term of (35) by integration by parts and Young’s inequality as follows:

$$\int (b \cdot \nabla) |j|^{q-2} j = -(q - 1) \int b \cdot \nabla |j|^{q-2} \Omega$$

We can absorb the diffusive term whereas for the second term of (37),

$$c \int |b|^2 |j|^{q-2} |\Omega|^2 \lesssim \|b\|_{L^q}^2 \|j\|_{L^q}^{q-2} |\Omega|_{L^s}^2$$

by Hölder’s and Gagliardo-Nirenberg inequalities, (34) and (4). Moreover, we can also write

$$\nu(q - 1) \int |\nabla j|^{q-2} = \frac{\nu(q - 1)}{q} \|\nabla |j|^{\frac{3}{2}}\|_{L^2}^2$$

On the other hand,

$$|j|_{L^{2q}} = \|j\|_{L^{\frac{q}{2}}}^{\frac{q}{2}} \lesssim \left( \|j\|_{L^{\frac{q}{2}}}^{\frac{q}{2}} \|\nabla |j|^{\frac{3}{2}}\|_{L^2}^{\frac{2}{3}} \right)^{\frac{q}{2}} \lesssim \|\nabla |j|^{\frac{3}{2}}\|_{L^2}^{\frac{2}{q} - 1}$$

by the Gagliardo-Nirenberg inequalities and Proposition 4.1 and therefore

$$|j|_{L^{2q}} \lesssim \|\nabla |j|^{\frac{3}{2}}\|_{L^2}^{\frac{q}{2} - 1}$$

Now we estimate the second integral in (35):

$$2 \int [\partial_t b_1 (\partial_t u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_t b_2 + \partial_2 b_1)] |j|^{q-2} j$$

and integrate in space to estimate

$$\sup_{t \in [0, T]} \|\Omega(t)\|_{L^s} \lesssim 1.$$
by Hölder’s inequalities, Lemma 2.1, (34), (40) and Young’s inequalities. Therefore, from (35)-(39), (41), after absorbing we obtain

$$\frac{1}{q} \partial_t \|j\|^q_{L^q} + \frac{\nu(q-1)}{4} \int |\nabla j| j^{q-2} |^2 + \frac{\nu(q-1)}{q^2} \|\nabla j\|^2_{L^2} \lesssim (1 + \|\nabla j\|^2_{L^2})(\|j\|^q_{L^q} + 1).$$

By Proposition 4.1, Gronwall’s inequality completes the proof of Proposition 4.2.

□

We now show that the criteria according to Proposition 3.2 is satisfied.

4.0.1. Bound of $\|w(t)\|_{L^\infty}$. Firstly, we take $L^2$-inner products on (7) with $-\Delta Z$ to obtain

$$\begin{aligned}
&\frac{1}{2} \partial_t \|\nabla Z\|^2_{L^2} + (\mu + \chi) \|\Delta Z\|^2_{L^2} + c_1 \|\nabla Z\|^2_{L^2} \\
= &\int (u \cdot \nabla) Z \Delta Z - c_2 \int w \Delta Z - \int (b \cdot \nabla) j \Delta Z \\
\lesssim &\|w\|_{L^p} \|\nabla Z\|_{L^2} \|\Delta Z\|_{L^2} + \|w\|_{L^2} \|\Delta Z\|_{L^2} + \|b\|_{L^\infty} \|\nabla j\|_{L^2} \|\Delta Z\|_{L^2} \\
\lesssim &\|w\|^2_{L^2} \|\nabla Z\|_{L^2} \|\Delta Z\|_{L^2} + \|\Delta Z\|_{L^2}^2 + \|b\|_{L^2} \|\nabla b\|_{L^2} \|\Delta Z\|_{L^2} \\
\leq &\frac{\mu + \chi}{2} \|\Delta Z\|^2_{L^2} + c(1 + \|\nabla Z\|^2_{L^2})(1 + \|\nabla j\|^2_{L^2})
\end{aligned}$$

by Hölder’s inequalities, (4), the Gagliardo-Nirenberg and Young’s inequalities and Proposition 4.2. After absorbing, by Gronwall’s inequality we obtain

$$\sup_{t \in [0,T]} \|\nabla Z(t)\|^2_{L^2} + \int_0^T \|\Delta Z\|^2_{L^2} dt \lesssim 1$$

(42)

due to Proposition 4.1. Next, we multiply (3b) by $|w|^{p-2} w$, integrate in space to obtain

$$\begin{aligned}
\frac{1}{p} \partial_t \|w\|^p_{L^p} + 2 \chi \|w\|^p_{L^p} \lesssim &\|\Omega\|_{L^\infty} \|w\|_{L^p}^{p-1} \lesssim \|Z\|_{L^p} \|w\|_{L^p}^p + \|w\|_{L^p}^p \\
\end{aligned}$$

by Hölder’s inequalities and (6). Dividing by $\|w\|_{L^p}^{p-1}$, we obtain

$$\|w(t)\|_{L^p} \lesssim \sup_{t \in [0,T]} \|w_0\|_{L^p} e^{bt} + \int_0^t e^{c(t-\tau)} \|Z\|_{L^p} d\tau \lesssim t + e^T \int_0^T \|Z\|_{L^p} d\tau.$$ \\

Taking limit $p \to \infty$, we obtain

$$\|w(t)\|_{L^\infty} \lesssim T + e^T \int_0^T \|Z\|_{L^\infty} d\tau.$$ \\

Therefore, by Proposition 4.1, (42) and Sobolev embedding, we obtain

$$\sup_{t \in [0,T]} \|w(t)\|_{L^\infty} \lesssim 1.$$ 

(43)

4.0.2. Bound of $\|\Omega(t)\|_{L^\infty}$. By (42), (43) and (6), we immediately obtain

$$\int_0^T \|\Omega(t)\|_{L^\infty} d\tau \lesssim \int_0^T \|Z(t)\|_{L^\infty} dt + \int_0^T \|w(t)\|_{L^\infty} dt \lesssim 1.$$ 

(44)
4.0.3. **Bound of** $\|\nabla j(t)\|_{L^2}$. **By Hölder’s inequality and Proposition 4.1,**

\[
\int_0^T \|\nabla j\|_{L^2} \, dt \leq \sqrt{T} \left( \int_0^T \|\nabla j\|_{L^2}^2 \, dt \right)^{\frac{1}{2}} \lesssim 1.
\]

This completes the proof of Theorem 1.1.

**References**


