ON THE NAVIER-STOKES EQUATIONS IN SCALING-INVARIANT SPACES IN ANY DIMENSION

KAZUO YAMAZAKI

Abstract. We study the Navier-Stokes equations with generalized dissipative term via a fractional Laplacian in any dimension higher than two. Using horizontal Biot-Savart law in the higher-dimensional case and anisotropic Littlewood-Paley theory with which we distinguish the first two directions and the rest, we obtain a blow-up criteria for its solution in norms which are invariant under the rescaling of these equations. The proof goes through for the classical Navier-Stokes equations if dimension is three, four or five. We also give heuristics and partial results toward further improvement.

Keywords: Anisotropic Littlewood-Paley theory; blow-up; Navier-Stokes equations; regularity.

1. Introduction, Statement of Main Results, Heuristics of the Proof

1.1. Introduction on regularity criteria and four-dimensional case. We study the following $N$-dimensional generalized Navier-Stokes equations (g-NSE):

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla \pi + \nu \Lambda^\alpha u &= 0, \\
\nabla \cdot u &= 0, \\
u(x,0) &= u_0(x),
\end{align*}
\]

where $u : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^N$, $\pi : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ are the fluid velocity and pressure fields respectively, $\nu > 0$ represents the viscosity and $\Lambda^\alpha$ with its exponent $\alpha \in \mathbb{R}^+$ is the fractional Laplacian defined via Fourier transform so that it has the Fourier multiplier of $m(\xi) = |\xi|^\alpha$; i.e. $\hat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$. Hereafter, without loss of generality we assume $\nu = 1$, we refer to the g-NSE at $\alpha = 1$ as the classical NSE and write $\partial_t$ for $\frac{\partial}{\partial t}$.

We emphasize already that our result allows the case $\alpha = 1$ and hence the classical NSE and although the fractional Laplacian gives the impression that its intent is purely a mathematical interest and not physical, in fact, it arises naturally in the study of equations in fluid mechanics and geophysics such as the two-dimensional surface quasi-geostrophic equations (see. e.g. equation (1) of [12]). Moreover, studying the generalized formulation with fractional Laplacian has allowed us to gain deeper understanding of many equations, e.g. the NSE ([31]),

---

12010MSC : 35B65, 35Q35, 35Q86
2Department of Mathematics and Statistics, Washington State University, Pullman, WA 99164-3113, U.S.A.
magneto-hydrodynamics (MHD) system ([32]), Boussinesq equations ([20]), Burgers equations ([22]), incompressible porous media equation governed by Darcy’s law ([7]).

In large, if $\alpha < \frac{1}{2} + \frac{N}{4}, N \geq 3$, the question of whether the solution with initial data sufficiently smooth, e.g. $u_0 \in H^s(\mathbb{R}^N)$ with $s \geq \frac{N}{2} + 1 - 2\alpha$, will remain in the same regularity class for all time remains unsolved. There are various ways to explain its reason, and here let us elaborate in terms of the known bounded quantity and rescaling. It is clear from the g-NSE (1) that if $u(x, t)$ is its solution, then so is $u_\lambda(x, t) \triangleq \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t), \lambda \in \mathbb{R}^+$ (if $\pi(x, t)$, then $\pi_\lambda(x, t) \triangleq \lambda^{4\alpha-2} \pi(\lambda x, \lambda^{2\alpha} t)$).

Now we may take $L^2(\mathbb{R}^N)$-inner products of (1a) with $u$, use the divergence-free condition from (1b) to obtain the kinetic energy and cumulative kinetic energy dissipation for any solution $u$ in a time interval $[0, T]$:

$$\sup_{t \in [0, T]} \|u(t)\|_{L^2_x}^2 + \int_0^T \|\Lambda^\alpha u\|_{L^2_x}^2 d\tau \leq \|u_0\|_{L^2_x}^2. \quad (2)$$

Taking the homogeneous Sobolev embedding of $H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-s}}(\mathbb{R}^N)$, assuming $\alpha < \frac{N}{2}$, we see that $\int_0^T \|u\|_{L^{\frac{2N}{N-s}}}^2 d\tau < \infty$ due to (2) if $u_0 \in L^2(\mathbb{R}^N)$ where $\|\cdot\|_{L^{\frac{2N}{N-s}}_x}$ is invariant under the rescaling of (1); i.e.

$$\int_0^T \|u_\lambda\|_{L^{\frac{2N}{N-s}}_x}^2 d\tau = \int_0^{\lambda^{2\alpha} T} \|u\|_{L^{\frac{2N}{N-s}}_x}^2 d\tau \quad \text{if and only if} \quad \alpha = \frac{1}{2} + \frac{N}{4}.$$ 

This computation is valid for $N = 2$ and informally it explains why the global well-posedness for the classical NSE, the g-NSE at $\alpha = 1$ so that $\alpha = \frac{1}{2} + \frac{N}{4}$, was achieved by Leray in [25]. In the words of those working in this direction of research, it is understood that the $N$-dimensional g-NSE is subcritical, critical, supercritical if $\alpha > \frac{1}{2} + \frac{N}{4}, \alpha = \frac{1}{2} + \frac{N}{4}, \alpha < \frac{1}{2} + \frac{N}{4}$ respectively, and in both the critical and subcritical cases, the positive result toward the global regularity issue is well-known (e.g. [32]).

In order to reach the resolution of the system (1) in dimensions higher than two, various results appeared, one of the most prominent being the so-called Serrin-type regularity criteria: if $u(x, t)$ is a weak solution to the $N$-dimensional classical NSE in $[0, T]$ and

$$u \in L^p_t L^p_x, \quad \frac{N}{p} + \frac{2}{r} \leq 1, \quad p \geq N, \quad (3)$$

then the solution is smooth in $[0, T]$ (see [30], [16] for the case $N = p = 3$, subsequently generalized to higher dimensions in [14]): the case of $\alpha \in [1, \frac{3}{2}], N = 3$ for the g-NSE (1) is shown in [41]. We remark that the norm $\|\cdot\|_{L^p_t L^p_x}$ is scaling-invariant precisely when $\frac{N}{p} + \frac{2}{r} = 1$ (and $\frac{N}{p} + \frac{2\alpha}{r} = 2\alpha - 1$ in case of the g-NSE (1)). We also mention a prominent work in [3] which states that if $u$ is a weak solution to the classical NSE in $[0, T]$ and

$$\nabla u \in L^p_t L^p_x, \quad \frac{N}{p} + \frac{2}{r} \leq 2, \quad r \in (1, \min\{2, \frac{N}{N-2}\}], \quad (4)$$

then $u$ is regular (see [41] for the case of the g-NSE (1)). Again, we emphasize that the norm $\|\nabla\cdot\|_{L^p_t L^p_x}$ is scaling-invariant precisely when $\frac{N}{p} + \frac{2}{r} = 2$.

We now mention some results in the effort to reduce the amount of components, $N$-many in (3) due to $u = (u^1, \ldots, u^N)$ and $N^2$-many in (4) due to $\nabla u =$
Navier-Stokes Equations in Scaling-Invariant Spaces

(∂_k u^j)_{1 \leq k, j \leq N}, on which conditions must be imposed. In particular, the authors in [23] proved that if u solves the three-dimensional classical NSE in [0, T] and

\[ u^3 \in L^p_T L^p_x, \quad \frac{3}{p} + \frac{2}{r} \leq \frac{5}{8}, \quad \frac{54}{23} \leq r \leq \frac{18}{5}, \]  

(5)

then it is smooth up to time T (see also [5]). We emphasize here that there is no condition imposed on \( u^1, u^2 \) and unfortunately the condition \( \frac{3}{p} + \frac{2}{r} < 1 \) disallows the scaling-invariant level. So many more results in this component reduction theory flourished, all of which we apologize for not being able to mention. Interestingly it is not impossible that the scaling-invariant level is actually kept. For example, due to the work of [2], it is well-known that the solution to the three-dimensional classical NSE has no blow-up in [0, T] if \( \nabla \times u \in L^1_T L^\infty_x \), where we note that the norm \( \| \nabla \times \cdot \|_{L^p_T L^p_x} \) is scaling-invariant. Subsequently, the authors in [8] showed that if \( u \) is a weak solution to the three-dimensional classical NSE in [0, T], and \( \sum_{k=1}^2 (\nabla \times u) \cdot e^k \in L^p_T L^p_x, \ \frac{3}{p} + \frac{2}{r} \leq 2, \ \frac{3}{2} < p < \infty \) where \( e^k \) is a standard basis element of \( \mathbb{R}^3 \), then \( u \) is a classical solution. Moreover, the authors in [24] obtained the following regularity criterion for the three-dimensional classical NSE:

\[ \partial_3 u \in L^p_T L^p_x, \quad \frac{3}{p} + \frac{2}{r} \leq 2, \ 2 \leq r \leq 3, \]  

(6)

for which the endpoint \( \frac{3}{p} + \frac{2}{r} = 2 \) allows the scaling-invariance. On the other hand, the results such as those of [6, 42] reduced components furthermore from \( \partial_3 u \) of (6) to \( \partial_3 u^3 \), but not at the scaling-invariant level. Finally, while whether the original result of Serrin for the three-dimensional classical NSE in (3) may be reduced to \( u^3 \) at the scaling-invariant level remains unknown, very recently Chemin and Zhang in [10] showed that if a blow-up occurs at \( T^* > 0 \), then

\[ \int_0^{T^*} \| u^3 \|^p_{L^p \dot{H}^{\frac{1}{2} + \frac{2}{p}}} \, d\tau = +\infty, \quad p \in (4, 6) \]  

(7)

where we emphasize that the norm \( \| \cdot \|_{L^p \dot{H}^{\frac{1}{2} + \frac{2}{p}}} \) is scaling-invariant; subsequently the range of \( p \) was improved in [11].

We now discuss the difficulty of extending these results to higher dimensions and in particular the mathematical significance of the four-dimensional case for the fluid equations such as (1). Heuristically, essentially due to Sobolev embedding, the study of the four-dimensional case leads to barely successful estimate if classical results but impossible if more recent results in a way that the range of parameters such as \( p, r \) in (6), (7) and many others in the needed estimates such as Propositions 5.1, 5.2, 5.3, 5.4 and 5.5 becomes empty. Since many decades ago, prominent mathematicians had already realized so, e.g. we quote “\( m = 4 \) is critical case and the proof would not work for \( m > 4 \)” by Kato in Section 4 [21]. The study of Scheffer in [29] is also focused strictly on the four-dimensional case, and that similar phenomenon of how certain methods in partial regularity theory work for the four-dimensional classical NSE (and six-dimensional stationary NSE) but not in any higher dimension is discussed in Remark 1.3 [13]: “four is the highest dimension in which we have such condition. In five or higher dimensional, such condition fails. Therefore, we cannot hope the existing methods work in five or higher dimensional case” (see also pg. 2212 [15]).
In terms of component reduction theory of the Serrin-type regularity criteria, in fact, every result that we have mentioned thus far, and all others in the literature to the best of the author’s knowledge, have been in the three-dimensional case, except the very recent work in [39] which in particular derived the following regularity criteria for the four-dimensional classical NSE:

\[ u^3, u^4 \in L^p_t L^r_x, \quad \frac{4}{r} + \frac{2}{p} \leq \frac{1}{p} + \frac{1}{2}, \quad 6 < p \leq \infty. \quad (8) \]

The difficulty of obtaining a result such as (8) because the four-dimensional case is essentially at the threshold, in a way that \( H^1(\mathbb{R}^N) \hookrightarrow L^N(\mathbb{R}^N) \) only for \( N \leq 4 \) but not for \( N > 4 \) and in fact \( \dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4) \), is described with much detail in the Introduction of [39], Section 1.2 and Remark 2.1 (2), (3) of [40]. Unfortunately this criteria in (8) is not at the scaling-invariant level, and it is expected that improving the upper bound of \( \frac{1}{p} + \frac{1}{2} \) in (8) back up to 1 is extremely difficult. We conclude this subsection by emphasizing that extension to the four-dimensional case being difficult, extending to even higher dimension is even harder.

1.2. Statement of main results. Throughout the rest of this manuscript, in any dimension \( N \geq 3 \), although not physical and arguably not universally acceptable, we shall refer to the first two directions as “horizontal” and the rest as “vertical.” This will make the notations significantly simpler. In particular, we write \( x = (x_h, x_v) \) where \( x_h \triangleq (x_1, x_2, 0, \ldots, 0), x_v \triangleq (0, 0, x_3, \ldots, x_N) \). Now we shall extend the notions of three-dimensional anisotropic Sobolev and Besov norms (e.g. [10]) to the \( N \)-dimensional case which has been studied by many for long time (e.g. pg. 51 [18], [19, 28]). We let \( S \) denote the Schwartz space and \( S' \) its dual.

**Definition 1.1.** For \( (s, s') \in \mathbb{R}^2 \), we let \( \dot{H}^{s, s'}(\mathbb{R}^N) = \{ f \in S' : \| f \|_{\dot{H}^{s, s'}} < \infty \} \)

where for \( \xi = (\xi_h, \xi_v), \xi_h = (\xi_1, \xi_2, 0, \ldots, 0), \xi_v = (0, 0, \xi_3, \ldots, \xi_N), \)

\[ \| f \|^2_{\dot{H}^{s, s'}} \triangleq \int_{\mathbb{R}^N} \left| \xi_h^s \xi_v^{s'} \hat{f}(\xi) \right|^2 \, d\xi < \infty. \]

We now present our main result where concerning the notations of Littlewood-Paley theory, we refer readers to the Section 2.

**Theorem 1.1.** Let the dimension \( N \in \mathbb{N}, N \geq 4 \), \( u_0 \in \dot{H}^{N-2\alpha} \mathbb{R}^N \) where

\[ \alpha \in \left[ \frac{N}{6} + \delta, \frac{N}{6} + \frac{2}{3} \right] \]

for any \( \delta > 0 \). Then there exists \( T^* > 0 \) such that on \([0, T^*)\), there exists a unique solution

\[ u \in C([0, T^*); H^{N-2\alpha} \mathbb{R}^N) \cap L^2((0, T^*); H^{N-\alpha} \mathbb{R}^N)) \]

to the \( g \)-NSE (1). Moreover, if for some \( \epsilon \in (0, \frac{N-2}{2}) \) and \( p \) such that

\[ \frac{2\alpha}{3\alpha - (\frac{N}{2})} < p \begin{cases} < \frac{4\alpha}{4\alpha - N} & \text{if } \alpha > \frac{N}{4}, \\ < +\infty & \text{if } \alpha = \frac{N}{4}, \\ \leq +\infty & \text{if } \alpha < \frac{N}{4}, \end{cases} \]

...
Let the dimension $\omega = \omega_{m}$. The anisotropic Sobolev space $\mathcal{H}_{\frac{N}{2}}$ is defined as

$$\mathcal{H}_{\frac{N}{2}} = \text{the homogeneous Sobolev embedding of } H^{\frac{N}{2}} \text{ into } L^p.$$  

(3)

where $\omega = \partial_{x}u^2 - \partial_{y}u^1$, then $T^* < \infty$.

Considering the range of $\alpha \geq \frac{N}{2} + \delta$, $\delta > 0$, we see that in the case dimension is four or five, we may take $\alpha = 1$ so that we wish to emphasize the following corollary that holds for the classicalNSE.

**Corollary 1.2.** Let the dimension $N = 4$ or 5, $u_0 \in \dot{H}^{\frac{N}{2}}(\mathbb{R}^N)$. Then there exists $T^* > 0$ such that on $[0, T^*)$, there exists a unique solution $u \in C([0, T^*]; \dot{H}^{\frac{N}{2}}(\mathbb{R}^N)) \cap L^2((0, T^*); \dot{H}^{\frac{N}{2}}(\mathbb{R}^N))$

to the classical NSE. Moreover, if for some $p$ such that

$$\frac{2}{3} - \left(\frac{\alpha}{2}\right) < p \begin{cases} < +\infty & \text{if } 1 = \frac{N}{4}, \\
\leq +\infty & \text{if } 1 < \frac{N}{4}, \end{cases}$$

$$\sum_{m=3}^{N} \int_{0}^{T^*} \| \omega^3(t) \|^{p(3-\frac{3}{p} - \frac{N}{2})}_{L^2} \int_{0}^{T^*} \| \Lambda^{\alpha} \omega^3 \|^{p(2+\frac{3}{p} + \frac{N}{2})}_{L^2} \, dt = \infty,$$

where $\omega^3 = \partial_{x}u^2 - \partial_{y}u^1$, then $T^* < \infty$.

**Remark 1.1.** (1) All the norms in (9), specifically

$$\int_{0}^{T} \| \Lambda^{\frac{N}{2} - \alpha} \partial_{m} u^m \|^{2}_{H_{\frac{N}{2} - \alpha}, dt},$$

$$\sum_{m=3}^{N} \int_{0}^{T} \| \Lambda^{\frac{N}{2} - \alpha} \partial_{m} u^m \|^{2}_{H_{\frac{N}{2} - \alpha}, dt},$$

$$\int_{0}^{T} \| \| \Lambda^{h} \Lambda^{\frac{N}{2} - \alpha} (\partial_{1}((-) \cdot e^2) - \partial_{2}((-) \cdot e^1)) \|_{L^2} \|^{2}_{L^2} dt,$$

$$\sup_{t \in [0, T]} \| \partial_{1}((-) \cdot e^2) - \partial_{2}((-) \cdot e^1) \|^{p(3-\frac{3}{p} - \frac{N}{2})}_{L^2}$$

$$\times \int_{0}^{T} \| \Lambda^{\alpha} (\partial_{1}((-) \cdot e^2) - \partial_{2}((-) \cdot e^1)) \|^{p(2+\frac{3}{p} + \frac{N}{2})}_{L^2} dt,$$

are invariant under the rescaling of (1). Moreover, the proof goes through in the threedimensional case but because the result in [10, 11] are better, we chose not to include such results in the statement of Theorem 1.1.

(2) We note that in [10] the authors needed $\nabla \times u|_{t=0} \in L^{2}(\mathbb{R})$ where by the homogeneous Sobolev embedding $W^{1,2}(\mathbb{R}^3) \to \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ despite the result in [17] which required only $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, while our initial data space $\dot{H}^{\frac{N}{2} + 1 - 2\alpha}(\mathbb{R}^N)$ is the critical space.

(3) The anisotropic Sobolev space $H^{-\gamma, \epsilon}$, $\epsilon \in (0, \frac{N-2}{2})$, may be considered as an analogue of $\mathcal{H}_{\theta} \triangleq \dot{H}^{-\gamma + \theta, \epsilon}$ for $\theta \in (0, \frac{1}{2})$ in [10].
(4) The results of [10, 11] have been generalized to the three-dimensional magneto-
hydrodynamics (MHD) system in [37, 26] respectively, and some general-
ization of Theorem 1.1 to the MHD system may be done as well.

1.3. Heuristic toward eliminating the condition of \( \omega^3 \) and \( \partial_m u^m, m = 3, \ldots N \) in (9). The purpose of this subsection is to discuss the idea of the proof of Theorem 1.1 and also the difficulty of eliminating the condition of \( \omega^3 \) and \( \partial_m u^m, m = 3, \ldots N \) to improve to a complete extension of the result in [10]. Firstly, one of the most crucial ingredients in the work of [10] was the following decomposition: for \( f = (f^1, f^2, f^3) \) such that \( \nabla \cdot f = 0, \Delta_h \triangleq \sum_{k=1}^2 \partial^2_k \), with \( e^3 = (0,0,1) \),

\[
(f^1, f^2, 0) = (-\partial_2, \partial_1, 0)\Delta_h^{-1} \nabla \times f \cdot e^3 - (\partial_1, \partial_2, 0)\Delta_h^{-1} \partial_3 f^3
\]

(see also [38]). In order to even start considering the higher-dimensional extension of the result in [10], it seemed that we will need to extend this identity and in particular \( \nabla \times u \cdot e^3 \) to an appropriate analogue in the higher-dimensional case. However, a cross product and a curl operator are meaningful at least physically and formally only in \( \mathbb{R}^3 \), and in the higher-dimensional case they are not so clear despite their importance, although we note that the curl of a vector field in any dimension is usually defined as an antisymmetric 2-tensor. Fortunately the following observation, of which its proof is straightforward via Fourier transform, was made in [40]:

Lemma 1.3. (Proposition 1.1, [40]) Suppose \( f = (f^1, \ldots, f^N) \in C^\infty(\mathbb{R}^N) \) such that \( \nabla \cdot f = 0 \). Under the notation of \( f^h \triangleq (f^1, f^2, 0, \ldots, 0), \nabla_h \triangleq (\partial_1, \partial_2, 0, \ldots, 0), \nabla^\perp_h \triangleq (-\partial_2, \partial_1, 0, \ldots, 0), \Delta_h = \sum_{k=1}^2 \partial^2_k \),

\[
f^h_{\text{curl}} \triangleq \nabla^\perp_h \Delta_h^{-1} (\partial_1 f^2 - \partial_2 f^1), \quad f^h_{\text{div}} \triangleq -\nabla_h \Delta_h^{-1} \sum_{k=3}^N \partial_k f^k,
\]

it holds that \( f^h = f^h_{\text{curl}} + f^h_{\text{div}} \).

Remark 1.2. To the best of the author’s knowledge, this identity in the three-
dimensional case was first used implicitly within a certain a priori estimate in [27]
(see (2.1) of [27]). We also remark that there is a discussion of higher-dimensional
curl operator on pg. 8 [9], although we failed to find any immediate application
toward an identity such as that in Lemma 1.3.

Now in the three-dimensional case, one can just take a curl operator on (1a)
and consider its third component. We cannot readily follow the same approach
due to the lack of precise formulation of a higher-dimensional curl operator. However,
thanks to Lemma 1.3, we do not necessarily have to take a curl operator in higher-
dimension but only need to estimate \( \omega^3 \triangleq \partial_1 u^2 - \partial_2 u^1 \). Thus, we apply \( \partial_1, \partial_2 \) on the second and first components of (1a) respectively to obtain

\[
\partial_1 (\partial_1 u^2) + \partial_1 ((u \cdot \nabla) u^2) + \partial_1 2\pi + \Lambda^{2\alpha} \partial_1 u^2 = 0,
\partial_2 (\partial_2 u^1) + \partial_2 ((u \cdot \nabla) u^1) + \partial_2 2\pi + \Lambda^{2\alpha} \partial_2 u^1 = 0.
\]

Now in the three-dimensional case, \( \nabla \pi \) disappears because \( \nabla \times (\nabla f) = 0 \) \forall scalar-valued function \( f \). In the higher-dimensional case, fortunately we may make use of the fact that \( \partial_1 2\pi = \partial_2 \pi \) and deduce

\[
\partial_1 \omega^3 + \partial_1 ((u \cdot \nabla) u^2) - \partial_2 ((u \cdot \nabla) u^1) + \Lambda^{2\alpha} \omega^3 = 0
\] (10)
where $\omega^3 = \partial_1 u^2 - \partial_2 u^1$. Moreover, we can rewrite
\begin{equation}
\partial_2((u \cdot \nabla)u^1) - \partial_1((u \cdot \nabla)u^2)
= (\partial_2 u \cdot \nabla)u^1 - (\partial_1 u \cdot \nabla)u^2 - (u \cdot \nabla)\omega^3
= \partial_2 u^1 \partial_1 u^1 + \partial_2 u^2 \partial_2 u^1 + \sum_{k=3}^{N} \partial_2 u^k \partial_k u^1
- \partial_1 u^1 \partial_2 u^1 - \partial_1 u^2 \partial_2 u^2 - \sum_{k=3}^{N} \partial_1 u^k \partial_k u^2 - (u \cdot \nabla)\omega^3
= - \partial_1 u^1 \omega^3 - \partial_2 u^2 \omega^3 + \left( \sum_{k=3}^{N} \partial_2 u^k \partial_k u^1 - \partial_1 u^k \partial_k u^2 \right) - (u \cdot \nabla)\omega^3
= \left( \sum_{k=3}^{N} \partial_k u^k \right) \omega^3 + \left( \sum_{k=3}^{N} \partial_2 u^k \partial_k u^1 - \partial_1 u^k \partial_k u^2 \right) - (u \cdot \nabla)\omega^3
\end{equation}
where we used divergence-free condition (1b). Applying (11) to (10) leads to
\begin{equation}
\partial_t \omega^3 + (u \cdot \nabla)\omega^3 - \Lambda^{2\alpha} \omega^3 = \sum_{k=3}^{N} \partial_k u^k \omega^3 + \sum_{k=3}^{N} \partial_2 u^k \partial_k u^1 - \partial_1 u^k \partial_k u^2.
\end{equation}
As we will see, e.g. in (40), (46), in addition to the bound on $\omega^3$, we will also wish to obtain a bound on $\partial_k u^k$, $k = 3, \ldots, N$. Now it is well-known that applying divergence operator on (1a) leads to $\pi = (-\Delta)^{-1} \sum_{k,m=1}^{N} \partial_k u^m \partial_m u^k$ due to the divergence-free property (1b). Thus, we may write from the g-NSE (1a), for $l \in \{3, \ldots, N\}$,
\begin{equation}
\partial_l u^l + (u \cdot \nabla)u^l + \Lambda^{2\alpha} u^l = \partial_t (-\Delta)^{-1} \sum_{k,m=1}^{N} \partial_k u^m \partial_m u^k.
\end{equation}
Applying $\partial_t$ to (13) leads to
\begin{equation}
\partial_t \partial_t u^l + \Lambda^{2\alpha} \partial_t u^l
= - (\partial_t u \cdot \nabla)u^l - (u \cdot \nabla)\partial_t u^l - \partial_t^2 (-\Delta)^{-1} \sum_{k,m=1}^{N} \partial_k u^m \partial_m u^k.
\end{equation}
We note that upon an attempt at the estimate of $\partial_k u^k$, $k = 3, \ldots, N$, one will see immediately that in contrast to the three-dimensional case, it will be necessary, e.g. in the four-dimensional case, to do additional estimates of the mixed terms, namely $\partial_t u^4, \partial_t u^3$, although this does not seem to cause much difficulty.

Now heuristically we can write the right hand side of (12) as
\begin{equation}
\sum_{k=3}^{N} \partial_k u^k \omega^3 + \partial_2 u^k \partial_k (u^1_{\text{curl}} + u^1_{\text{div}}) - \partial_1 u^k \partial_k (u^2_{\text{curl}} + u^2_{\text{div}})
\end{equation}
due to Lemma 1.3. As $u^1_{\text{curl}}, u^2_{\text{curl}}$ consist of $\omega^3$, we see that $\partial_k u^k \omega^3, \partial_2 u^k \partial_k u^1_{\text{curl}}$ and $\partial_1 u^k \partial_k u^2_{\text{curl}}$ in (15) are linear in $\omega^3$ while because $u^1_{\text{div}}, u^2_{\text{div}}$ do not consist of $\omega^3$, we may consider $\partial_2 u^k \partial_k u^1_{\text{div}}, \partial_1 u^k \partial_k u^2_{\text{div}}$ to be just forcing terms in the time
evolution equation of $\omega^3$ (12). On the other hand, in (14),
\[- \partial_t^2 (-\Delta)^{-1} \sum_{k,m=1}^N \partial_k u_m \partial_m u^k \]
\[= - \partial_t^2 (-\Delta)^{-1} \left[ \sum_{k,m=1}^N \partial_k u_m \partial_m u^k + \sum_{k=1}^N \sum_{m=3}^N \partial_k u_m \partial_m u^k + \sum_{k=3}^N \sum_{m=1}^N \partial_k u_m \partial_m u^k \right] \]
where due to Lemma 1.3 we have
\[\sum_{k,m=1}^2 \partial_k u_m \partial_m u^k = \sum_{k,m=1}^2 \partial_k (u_{\text{curl}}^m + u_{\text{div}}^m) \partial_m (u_{\text{curl}}^k + u_{\text{div}}^k) \]
of which because $u_{\text{curl}}^m, u_{\text{curl}}^k$ consist of $\omega^3$, the quadratic terms here are $\partial_k u_{\text{curl}}^m \partial_m u_{\text{curl}}^k$. This heuristic gives the impression that perhaps the estimate involving $\omega^3$ is somewhat easier than that involving $\partial_k u^k, k = 3, \ldots, N$. We explain in the Appendix that even in the four-dimensional case, the estimate involving the former seems very difficult. We conclude this discussion by noting that if an estimate of $\|\omega^3(t)\|_{L^2}$ may be completed, it will immediately allow us to eliminate the condition on $u$ in (9) using Hölder’s inequality.

The structure of the proof of Theorem 1.1 is as follows. Firstly, by Theorem 6.2 [34] (see also [33]), the existence and uniqueness of the local solution $u \in C([0, T); \dot{H}^{\sigma + 1 - 2\alpha}(\mathbb{R}^N)) \cap L^2((0, T); \dot{H}^{\sigma + 1 - \alpha}(\mathbb{R}^N))$ follows. We will show that the converse of (9) implies $\sum_{k,l=1}^N \int_0^T \|\partial_k u^k\|_{\dot{B}^{\alpha,1}_{\infty,1}} < \infty$ in Section 4 which in turn implies that $u$ remains in the regularity of $C([0, T + \delta]; \dot{H}^{\sigma + 1 - 2\alpha}(\mathbb{R}^N)) \cap L^2((0, T + \delta); \dot{H}^{\sigma + 1 - \alpha}(\mathbb{R}^N))$ for some $\delta > 0$ by Proposition 3.1 and standard continuation of local theory argument.

2. Preliminaries

2.1. Definitions, Notations and Past Results. We write $A \lesssim_{a,b} B, A \approx_{a,b} B$ when there exists a constant $c \geq 0$ of no significant dependence except on $a, b$ such that $A \leq c B, A = c B$ respectively. For simplicity, we often write $\int f$ to denote $\int_{\mathbb{R}^N} f \, dx$. We recall the mixed Lebesgue spaces from [4], reminding ourselves that its order matters; i.e.
\[\left\| \| f(\cdot, x_2) \|_{L^{p_1}(X_1, \mu_1)} \|_{L^{p_2}(X_2, \mu_2)} \right\|_{L^{p_1}(X_1, \mu_1)} \leq \left\| \| f(x_1, \cdot) \|_{L^{p_2}(X_2, \mu_2)} \|_{L^{p_1}(X_1, \mu_1)} \right\|_{L^{p_1}(X_1, \mu_1)} \]
for any two measure spaces $(X_1, \mu_1), (X_2, \mu_2)$ with $1 \leq p_1 \leq p_2 \leq \infty$. We also recall the Littlewood-Paley decomposition: with $\chi, \phi$ smooth functions such that
\[
\text{supp } \phi \subset \{ \zeta \in \mathbb{R} : \frac{3}{4} \leq |\zeta| \leq \frac{8}{3} \}, \quad \sum_{j \in \mathbb{Z}} \phi(2^{-j} \zeta) = 1, \\
\text{supp } \chi \subset \{ \zeta \in \mathbb{R} : |\zeta| \leq \frac{4}{3} \}, \quad \chi(\zeta) + \sum_{j \geq 0} \phi(2^{-j} \zeta) = 1,
\]
we denote the classical homogeneous and nonhomogeneous Littlewood-Paley operators for \( \xi = (\xi_h, \xi_v) \in \mathbb{R}^N \),
\[
\hat{\Delta}_j f \triangleq \mathcal{F}^{-1}((\phi(2^{-j}|\xi|))\hat{f}), \quad \hat{\dot{S}}_j f \triangleq \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\hat{f}),
\]
(17)
\[
\Delta_j f \triangleq \begin{cases} 
0 & \text{if } j \leq -2, \\
\mathcal{F}^{-1}(\chi(|\xi|)\hat{f}) & \text{if } j = -1, \\
\hat{\Delta}_j f & \text{if } j \geq 0,
\end{cases}
\]
and the anisotropic case similarly:
\[
\hat{\Delta}^h_j f \triangleq \mathcal{F}^{-1}((\phi(2^{-k}|\xi_h|))\hat{f}), \quad \hat{\dot{S}}^h_j f \triangleq \mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\hat{f}),
\]
(19)
\[
\hat{\Delta}^v_j f \triangleq \mathcal{F}^{-1}((\phi(2^{-l}|\xi_v|))\hat{f}), \quad \hat{\dot{S}}^v_j f \triangleq \mathcal{F}^{-1}(\chi(2^{-l}|\xi_v|)\hat{f}),
\]
(20)
with
\[
\Delta^h_k f \triangleq \begin{cases} 
0 & \text{if } k \leq -2, \\
\mathcal{F}^{-1}(\chi(|\xi_h|)\hat{f}) & \text{if } k = -1, \\
\hat{\Delta}^h_k f & \text{if } k \geq 0,
\end{cases}
\]
(21)
\[\Delta^v_l f \triangleq \begin{cases} 
0 & \text{if } l \leq -2, \\
\mathcal{F}^{-1}(\chi(|\xi_v|)\hat{f}) & \text{if } l = -1, \\
\hat{\Delta}^v_l f & \text{if } l \geq 0.
\end{cases}
\]
We define \( S^t_h \) to be the subspace of \( S^t \) such that every \( f \in S^t_h \) satisfies \( \lim_{j \to -\infty} ||\hat{\dot{S}}_j f||_{L^\infty} = 0 \).

**Definition 2.1.** For \( p, q, s \in [1, \infty], s \in \mathbb{R}, s < \frac{N}{p} (s = \frac{N}{p} \text{ if } q = 1) \), we define the Besov spaces \( \dot{B}^{s}_{p,q}(\mathbb{R}^N) \triangleq \{ f \in S' : ||f||_{\dot{B}^{s}_{p,q}} < \infty \} \) where
\[
||f||_{\dot{B}^{s}_{p,q}} \triangleq \left( \sum_{j \in \mathbb{Z}} (2^{js}||\hat{\dot{S}}_j f||_{L^p})^q \right)^{\frac{1}{q}}.
\]
(22)
Moreover, for \( p \in (1, \infty) \), we shall use the notations \( \mathcal{B}_p \triangleq \dot{B}^{-2\alpha+\frac{2N}{p}}_{\infty,\infty} \).

We define the anisotropic Besov spaces \( (\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})_v \) as the space of distributions in \( S^t_h \) endowed with its norm of
\[
||f||_{(\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})_v} \triangleq \left( \sum_{k \in \mathbb{Z}} 2^{q_1k s_1} \left( \sum_{l \in \mathbb{Z}} 2^{q_2 l s_2} ||\hat{\Delta}^h_k \hat{\Delta}^v_l f||_{L^p} \right)^q \right)^{\frac{1}{q_1}}.
\]
(23)
It is well-known that \( \dot{B}^{2}_{2,2} = H^s \) (cf. [1]). Moreover, the special case of the anisotropic Besov spaces recovers the anisotropic Sobolev spaces: \( (\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})_v |_{p=q_1=q_2=2} = H^{s_1,s_2} \). We recall the important Bony’s para-product decomposition:
\[
f g = T(f,g) + T(g,f) + R(f,g)
\]
(24)
where
\[
T(f,g) \triangleq \sum_{j \in \mathbb{Z}} \hat{\dot{S}}_{j-1} f \hat{\Delta}_j g, \quad R(f,g) \triangleq \sum_{j \in \mathbb{Z}} \hat{\Delta}_{j+1} f \hat{\Delta}_j g, \quad \text{where } \hat{\Delta}_j \triangleq \sum_{l=j-1}^{j+1} \hat{\Delta}_l
\]
(25)
(see e.g. [9]). We also recall the useful anisotropic Bernstein’s inequalities:

**Lemma 2.1.** Let \( B_h \) (resp. \( B_v \)) a ball in \( \mathbb{R}^2 \) (resp. \( \mathbb{R}^{-2} \)) and \( C_h \) (resp. \( C_v \)) a ring in \( \mathbb{R}^2 \) (resp. \( \mathbb{R}^{-2} \)), \( \nabla_h = (\partial_1, \partial_2, 0, \ldots, 0), \nabla_v = (0, 0, \partial_3, \ldots, \partial_N) \).
Moreover, let $1 \leq p_2 \leq \infty$, $1 \leq q_2 \leq q_1 \leq \infty$. Then
\[
\|\nabla \hat{f}\|_{L^{p_2}_t(L^{q_2}_x)} \lesssim 2^{k(|\alpha|+2(N-2)(\frac{1}{p_2} - \frac{1}{p_1}))}\|f\|_{L^{p_1}_t(L^{q_1}_x)} \quad \text{if } \text{supp} \hat{f} \subset 2^kB_h,
\]
\[
\|\nabla^\beta \hat{f}\|_{L^{p_2}_t(L^{q_2}_x)} \lesssim 2^{(|\beta|+(N-2)(\frac{1}{p_2} - \frac{1}{p_1}))}\|f\|_{L^{p_1}_t(L^{q_1}_x)} \quad \text{if } \text{supp} \hat{f} \subset 2^kB_v,
\]
\[
\|f\|_{L^{p_2}_t(L^{q_2}_x)} \lesssim 2^{-kM} \sup_{|\alpha|=M} \|\nabla^\alpha \hat{f}\|_{L^{p_1}_t(L^{q_1}_x)} \quad \text{if } \text{supp} \hat{f} \subset 2^kB_h,
\]
\[
\|f\|_{L^{p_2}_t(L^{q_2}_x)} \lesssim 2^{-lM} \sup_{|\beta|=M} \|\nabla^\beta \hat{f}\|_{L^{p_1}_t(L^{q_1}_x)} \quad \text{if } \text{supp} \hat{f} \subset 2^lC_v.
\]

**Remark 2.1.** Let us remark on the complexity of the anisotropic Littlewood-Paley theory. E.g. although by Bernstein’s inequality we have
\[
\|\hat{\Delta}_j \partial_2 f\|_{L^p} \lessgtr \|\hat{\Delta}_j \nabla f\|_{L^p} \lesssim 2^j \|\hat{\Delta}_j f\|_{L^p},
\]
bounding $\partial_2$ by $\nabla$ seems not optimal; hence, one might choose to estimate by
\[
\|\hat{\Delta}_j \partial_2 f\|_{L^p} = \|\hat{\Delta}_j \sum_{k \in \mathbb{Z}} \hat{\Delta}_k^h \partial_2 f\|_{L^p} \lesssim \sum_{k \in \mathbb{Z}} \|\hat{\Delta}_j \hat{\Delta}_k^h \partial_2 f\|_{L^p} \lesssim \sum_{k \in \mathbb{Z}} 2^k \|\hat{\Delta}_j \hat{\Delta}_k^h f\|_{L^p}.
\]
The difficulty here is now the bi-infinite sum over $k$, which leads to anisotropic Besov spaces, from which going back to the classical Besov spaces $\dot{B}^{p,q}_s$ requires several conditions, well described in the hypothesis of Propositions 5.1, 5.2. Indeed, while if $p = p_1, p_2$, then $\|f\|_{L^{p_1}_t(L^{q_1}_x)} = \|f\|_{L^p}$, even if $q_1 = q_2$,
\[
\|f\|_{\dot{B}^{s_1,s_2}_{p_1,q_1}(\mathbb{R}^N)} \neq \left( \sum_{j \in \mathbb{Z}} 2^{(s_1+s_2)j} \|\hat{\Delta}_j f\|_{L^p}^q \right)^{\frac{1}{q}} = \|f\|_{\dot{B}^{s_1+s_2}_{p,q}(\mathbb{R}^N)}.
\]

Finally, for convenience, let us recall the following general partial sum formula as we will rely on it frequently:

**Lemma 2.2.** For any $r \in \mathbb{C} \setminus \{1\}$, not necessarily requiring that $|r| < 1$, $n, m \in \mathbb{Z}$,
\[
\sum_{j=n}^{m} r^j = \frac{r^{m+1} - r^n}{r - 1}.
\]

3. **Preliminary blow-up criterion**

We now prove a preliminary blow-up criterion, which is a kind of an extension of Lemma 8.1 [10], Proposition 4.1 in [37]. In particular, for generality we prove the version for $H^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^N)$ instead of $H^1(\mathbb{R}^3)$.

**Proposition 3.1.** Suppose $u$ is a smooth solution for the $q$-NSE (1) in case $N \in \mathbb{N}, N \geq 3, \alpha \in \left[\frac{N}{6} + \delta, \frac{N}{6} + \frac{3}{2}\right], \delta > 0$, over time interval $[0,T]$ starting from $u_0 \in H^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^N)$. Then
\[
\sup_{t \in [0,T]} \|u(t)\|_{H^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^N)}^2 + \int_0^T \|\Lambda^\alpha u\|_{H^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^N)}^2 \, d\tau \\
\lesssim e^{\sum_{k,l=1}^N \int_0^T \|\partial_k u^k\|_{\mathcal{B}_{p_k,l}} \, d\sigma} \|u_0\|_{H^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^N)}^2,
\]
where $\mathcal{B}_{p_k,l} = \dot{B}_{\infty,\infty}^{-2\alpha+\frac{2\alpha}{p_k,l}}, p_k,l \in (1,\infty)$.
Remark 3.1. In contrast to the typical blow-up criterion, the key feature of this estimate is that it allows us to choose different $p_{k,l}, k, l \in \{1, \ldots, N\}$; we will see that this is crucial in (39), (40), (46).

Proof. We apply $\hat{\Delta}_j$ on the g-NSE (1a), take $L^2$-inner products with $\hat{\Delta}_j u$ to obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\hat{\Delta}_j u\|_{L^2}^2 + \|\hat{\Delta}_j \Lambda^\alpha u\|_{L^2}^2 = -\int \hat{\Delta}_j ((u \cdot \nabla) u) \cdot \hat{\Delta}_j u. \tag{25}
\end{equation}
We multiply by $2^{2j(\frac{N}{2}+1-2\alpha)}$, sum over $j \in \mathbb{Z}$ to compute
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|u\|_{H^{\frac{N}{2}+1-2\alpha}}^2 + \|\Lambda^\alpha u\|_{H^{\frac{N}{2}+1-2\alpha}}^2 = -\int ((u \cdot \nabla) u | u)_{H^{\frac{N}{2}+1-2\alpha}} \tag{26}
\end{equation}
where we may use Bony’s paraproducts (23) to write
\begin{equation}
\begin{aligned}
\|(u \partial_t u^k | u^k)_{H^{\frac{N}{2}+1-2\alpha}} &\leq \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \|\hat{\Delta}_j T(u^l, \partial_t u^k) | \hat{\Delta}_j u^k)\| \\
&+ \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \|\hat{\Delta}_j T(\partial_t u^k, u^l) | \hat{\Delta}_j u^k)\| \\
&+ \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \|\hat{\Delta}_j R(u^l, \partial_t u^k) | \hat{\Delta}_j u^k)\| \triangleq I_1 + I_2 + I_3.
\end{aligned} \tag{27}
\end{equation}
We start with
\begin{equation}
I_1 = \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \|\hat{\Delta}_j (\sum_{j':|j-j'| \leq 4} \hat{S}_{j'-1} u^l \hat{\Delta}_j \partial_t u^k | \hat{\Delta}_j u^k)\| \\
\leq \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \int \hat{S}_{j-1} u^l \hat{\Delta}_j \partial_t u^k \hat{\Delta}_j u^k | \\
+ 2^{2j(\frac{N}{2}+1-2\alpha)} \int \sum_{j':|j-j'| \leq 4} \hat{\Delta}_j \hat{S}_{j'-1} u^l \hat{\Delta}_j \partial_t u^k \hat{\Delta}_j u^k | \\
+ 2^{2j(\frac{N}{2}+1-2\alpha)} \int \sum_{j':|j-j'| \leq 4} (\hat{S}_{j'-1} u^l - \hat{S}_{j-1} u^l) \hat{\Delta}_j \hat{\Delta}_j \partial_t u^k \hat{\Delta}_j u^k | \\
\triangleq I_{1,1} + I_{1,2} + I_{1,3} \leq \sum_{j} 2^{2j(\frac{N}{2}+1-2\alpha)} \int \frac{1}{2} \hat{S}_{j-1} u^l \partial_t (\hat{\Delta}_j u^k)^2 = 0. \tag{29}
\end{equation}
Next, from (28)
\begin{equation}
I_{1,2} \leq \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \sum_{j':|j-j'| \leq 4} \|\hat{\Delta}_j \hat{\Delta}_j \partial_t u^k \|_{L^2} \|\hat{\Delta}_j u^k\|_{L^2} \\
\leq \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} 2^{-j} \sum_{j':|j-j'| \leq 4} \|\nabla \hat{S}_{j'-1} u^l\|_{L^\infty} \|\hat{\Delta}_j \partial_t u^k\|_{L^2} \|\hat{\Delta}_j u^k\|_{L^2} \\
\lesssim \sum_j \sum_{j'=1}^N 2^{2(N+1-4\alpha)} \|\partial_t \hat{S}_{j-1} u^l\|_{L^\infty} \|\hat{\Delta}_j \partial_t u^k\|_{L^2} \|\hat{\Delta}_j u^k\|_{L^2}.
\end{equation}
where we used Hölder’s inequality, the well-known commutator estimate (see Lemma 2.1 [35], and also Lemma 2.97 [1]) and that we may assume \( j = j' \) when \(|j - j'| \leq 4\) modifying constants appropriately. Furthermore, we continue to bound by

\[
I_{1,2} \lesssim \sum_{l' = 1}^{N} \sum_{j} 2^{j(N+1-2\alpha - \frac{2\alpha}{p_{l},p'})} \sum_{j' \leq j - 2} 2^{j'(-2\alpha + \frac{2\alpha}{p_{l},p'})} 2^{j'(-\alpha - 2\alpha - \frac{2\alpha}{p_{l},p'})} \left\| \tilde{\Delta}_{j'} \partial_{j'} u' \right\|_{L^{\infty}} \left\| \tilde{\Delta}_{j} \partial_{j} u \right\|_{L^{2}} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}}
\]

\[
\lesssim \sum_{l' = 1}^{N} \sum_{j} 2^{j(N+2-2\alpha - \frac{2\alpha}{p_{l},p'})} \left\| \partial_{j'} u' \right\|_{B_{p_{l},1}} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}}
\]

\[
\approx \sum_{l' = 1}^{N} \left\| \partial_{j'} u' \right\|_{B_{p_{l},1}} \sum_{j} \left( 2^{j(N+1-2\alpha)} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}} \right)^{\frac{2}{p_{l},p'}} \times \left( 2^{j(N+1-2\alpha)} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}} \right)^{2(1 - \frac{1}{p_{l},p'})}
\]

by (17), Young’s inequality for convolution, the hypothesis that \( p_{k,l} \in (1, \infty) \) so that \( 2\alpha - \frac{2\alpha}{p_{l},p'} > 0 \) and Hölder’s inequality. Next, from (28)

\[
I_{1,3} \lesssim \sum_{j} 2^{2j\left( \frac{3}{2} \alpha + 1 - 2\alpha \right)} \sum_{j': |j - j'| \leq 4} \left\| (\tilde{\Delta}_{j'} - \tilde{\Delta}_{j} u^{k}) \right\|_{L^{2}} \left\| \tilde{\Delta}_{j} \partial_{j} u \right\|_{L^{2}} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}}
\]

\[
\lesssim \sum_{j} 2^{2j\left( \frac{3}{2} \alpha + 1 - 2\alpha \right)} \sum_{j': |j - j'| \leq 4} \sum_{j''} \left\| \tilde{\Delta}_{j''} u' \right\|_{L^{2}} \left\| \tilde{\Delta}_{j} \tilde{\Delta}_{j''} \partial_{j''} u' \right\|_{L^{2}} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}}
\]

by Hölder’s inequality. Now we may follow the approach of [10] to write

\[
\tilde{\Delta}_{j'} u' \approx \sum_{l' = 1}^{N} \tilde{\Delta}_{j'}^{l'} \tilde{\Delta}_{j'} u' \approx \sum_{l' = 1}^{N} 2^{-j' \alpha} \tilde{\Delta}_{j'}^{l'} \tilde{\Delta}_{j'} \partial_{j'} u' \tag{31}
\]

(see (8.3) [10] for the definition of \( \tilde{\Delta}_{j'}^{l'} \)) and continue our estimate by

\[
I_{1,3} \lesssim \sum_{j} 2^{2j\left( \frac{3}{2} \alpha + 1 - 2\alpha \right)} \sum_{j': |j - j'| \leq 4} \sum_{j''} \left( 2^{-j'' \alpha} \left\| \tilde{\Delta}_{j''} u' \right\|_{L^{2}} \right)^{\frac{2}{p_{l},p'}} \times \left( 2^{j(N+1-2\alpha)} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}} \right)^{2(1 - \frac{1}{p_{l},p'})}
\]

\[
\lesssim \sum_{l' = 1}^{N} \left\| \partial_{j'} u' \right\|_{B_{p_{l},1}} \sum_{j} \left( 2^{j\left( \frac{3}{2} \alpha + 1 - 2\alpha \right)} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}} \right)^{\frac{2}{p_{l},p'}} \times \left( 2^{j(N+1-2\alpha)} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}} \right)^{2(1 - \frac{1}{p_{l},p'})}
\]

\[
\lesssim \sum_{l' = 1}^{N} \left\| \partial_{j'} u' \right\|_{B_{p_{l},1}} \left\| u \right\|_{B_{p_{l},1}}^{2(1 - \frac{1}{p_{l},p'})} \left\| \tilde{\Delta}_{j} u^{k} \right\|_{L^{2}}^{2(1 - \frac{1}{p_{l},p'})}
\]
where we used that we may assume \( j = j' \) when \(|j - j'| \leq 4\) modifying constants appropriately. Plancherel theorem and Hölder’s inequality. Thus, we have shown due to (29), (30), (31) applied to (28),

\[
I_1 \leq I_{1,1} + I_{1,2} + I_{1,3} \lesssim \sum_{I' = 1}^{N} \| \partial_t u' \|_{B_{p_k,1}} \| u \|_{\dot{H}^{k}}^2 \| \Lambda^\alpha u \|_{\dot{H}^{k + \frac{3}{5} + \frac{3}{2} \alpha}}^{2(1 - \frac{1}{p_k})}. \tag{32}
\]

Next, from (27)

\[
I_2 \lesssim \sum_j 2^{2j\left(\frac{k}{2} - \frac{1}{2} - 2\alpha\right)} \| \dot{\Delta}_j \sum_{j'} \dot{\Delta}_{j'-1} \partial_t u' \Delta_{j'} u' \|_{L^2} \| \Delta_j u^k \|_{L^2},
\]

\[
\lesssim \sum_j 2^{2j\left(\frac{k}{2} - \frac{1}{2} - 2\alpha\right)} \| \dot{\Delta}_j \sum_{j'|\ |j-j'| \leq 4} \dot{\Delta}_{j'-1} \partial_t u' \|_{L^\infty} \| \Delta_j u' \|_{L^2} \| \Delta_j u^k \|_{L^2},
\]

\[
\lesssim \sum_j 2^{2j\left(\frac{k}{2} - \frac{1}{2} - 2\alpha\right)} \| \dot{\Delta}_j \partial_t u' \|_{L^\infty} \| \Delta_j u' \|_{L^2} \| \Delta_j u^k \|_{L^2}
\]

where we used (24), Hölder’s inequality and that we may assume \( j' = j \) when \(|j - j'| \leq 4\) by modifying constants. We furthermore continue to bound by

\[
I_2 \lesssim \sum_j 2^{2j\left(\frac{k}{2} - \frac{1}{2} - 2\alpha\right)} \sum_{j'\ : \ j' \leq j - 2} \| \dot{\Delta}_{j'} \partial_t u' \|_{L^\infty} \| \Delta_j u' \|_{L^2} \| \Delta_j u^k \|_{L^2}, \tag{33}
\]

\[
\approx \sum_j 2^{j(N - 2 - 2\alpha - \frac{2\alpha}{p_k - 1})} \sum_{j'\ : \ j' \leq j - 2} 2^{j' - j}(2^\alpha - \frac{2\alpha}{p_k - 1}) 2^{j' - (2\alpha + \frac{2\alpha}{p_k - 1})} \times \| \dot{\Delta}_{j'} \partial_t u' \|_{L^\infty} \| \Delta_j u' \|_{L^2} \| \Delta_j u^k \|_{L^2}
\]

\[
\lesssim \sum_j 2^{2j(N - 2 - 2\alpha - \frac{2\alpha}{p_k - 1})} \| \partial_t u' \|_{B_{p_k,1}} \| \Delta_j u' \|_{L^2} \| \Delta_j u^k \|_{L^2}
\]

\[
\approx \| \partial_t u' \|_{B_{p_k,1}} \sum_j (2^{j\left(\frac{k}{2} - \frac{1}{2} - 2\alpha\right)} \| \Delta_j u \|_{L^2}) \frac{2\alpha}{p_k - 1} 2^{j\left(\frac{k}{2} - \frac{1}{2} - 2\alpha\right)} \| \Delta_j \Lambda^\alpha u \|_{L^2}^{2(1 - \frac{1}{p_k})} \]

\[
\lesssim \| \partial_t u' \|_{B_{p_k,1}} \| u \|_{\dot{H}^{k}}^2 \| \Lambda^\alpha u \|_{\dot{H}^{k + \frac{3}{5} + \frac{3}{2} \alpha}}^{2(1 - \frac{1}{p_k})}
\]

by Young’s inequality for convolution, the hypothesis that \( p_k > 1 \), Plancherel theorem, and Hölder’s inequality. Finally, from (27) we may rewrite

\[
I_3 = \sum_j 2^{2j\left(\frac{k}{2} - \frac{1}{2} - 2\alpha\right)} \| \dot{\Delta}_j \sum_{j'\ : \ j' \geq j - \delta} \partial_t (\dot{\Delta}_{j'} u' \dot{\Delta}_{j'} u^k) \|_{B_{p_k,1}} \tag{34}
\]

for some \( \delta \in \mathbb{Z}^+ \) due to (24) and (1b). Now we write

\[
\dot{\Delta}_j u' \approx \sum_{I' = 1}^{N} \dot{\Delta}_{j'} \dot{\Delta}_j u' \approx \sum_{I' = 1}^{N} 2^{-j'} \dot{\Delta}_{j'} \dot{\Delta}_j \partial_t u',
\]
which is needed to obtain $2^{j-j'}$ as we shall subsequently see, and compute

$$I_3 \approx \sum_j 2^{2j} (\frac{N}{2} + 1 - 2\alpha) \left( \frac{\hat{\Lambda}_j}{2} \sum_{j', j' \geq j - \delta} \partial_t \left( \sum_{l' = 1}^N 2^{-j'} \hat{\Lambda}_{j'} \hat{\Delta}_{j'} \partial_{l'} u^{k} \hat{\Delta}_{j'} u^{k} \right) \right) \left( \hat{\Delta}_j u^{k} \right) \right)$$

$$\leq \sum_j 2^{2j} (\frac{N}{2} + 1 - 2\alpha) \sum_{j', j' \geq j - \delta} 2^{-j'} \left( \frac{\hat{\Lambda}_{j'} \hat{\Delta}_{j'} \partial_{l'} u^{k} \hat{\Delta}_{j'} u^{k} \right) \left( \frac{\hat{\Delta}_j u^{k}}{L} \left( \hat{\Delta}_j u^{k} \right) \right)$$

where we used Hölder’s inequality, Bernstein’ inequality, Young’s inequality for convolution, and Flancherel theorem. The powers must be distributed differently from the previous terms such as (30), (31) and (33) here. We compute

$$\sum_j 2^{2j} (\frac{N}{2} + 1 - 2\alpha) \sum_{j', j' \geq j - \delta} 2^{-j'} \left( \frac{\hat{\Lambda}_{j'} \hat{\Delta}_{j'} \partial_{l'} u^{k} \hat{\Delta}_{j'} u^{k} \right) \left( \frac{\hat{\Delta}_j u^{k}}{L} \left( \hat{\Delta}_j u^{k} \right) \right)$$

by Hölder’s inequality, Young’s inequality for convolution and that $\frac{N}{2} + 2 - 3\alpha + \frac{\alpha}{p_{l', i'}} > 0$ due to the range of $\alpha$. Thus, in consideration of (32), (33), (34), (35), (36) in (27), we have shown

$$\frac{1}{2} \partial_t \| u \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}} + \| \Lambda^\alpha u \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}}$$

$$\leq \sum_{k, l = 1}^N \| \partial_t u^k \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}} \| \Lambda^\alpha u \|^{2(1 - \frac{1}{p_{l', i'}})}$$

$$\leq \frac{1}{2} \| \Lambda^\alpha u \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}} + c \sum_{k, l = 1}^N \| \partial_t u^k \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}} \| u \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}}$$

$$\leq \frac{1}{2} \| \Lambda^\alpha u \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}} + c \sum_{k, l = 1}^N \| \partial_t u^k \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}} \| u \|^{\frac{2}{\frac{N}{2} + 1 - 2\alpha}}$$
by Young’s inequality. Subtracting $\frac{1}{2} \| \Lambda^\alpha u \|_{L^2}^2 \lesssim_{\alpha} 1$ from both sides and applying Gronwall’s inequality complete the proof of Proposition 3.1. 

4. Verifying the blow-up criteria 

Due to Proposition 3.1, it suffices to show $\sum_{k,l=1}^N \int_0^T \| \partial_t u^k \|_{B_{p,k,l}^{(\alpha, k, l)}}^p \, dt \lesssim 1$ for some $p, k, l \in (1, \infty)$ assuming 

$$\sum_{m=3}^N \int_0^T \| u^m \|_{B_{p,k,l}^{\alpha}}^p \, dt + \| \Lambda^\frac{3}{2} \partial_m u^m \|_{L^2}^2 + \| \left( \| \Delta_k^p \Lambda^\alpha \|_{L^2} \right)^{\frac{p}{2}} \| \partial_t \|_{L^2}^p \, dt \lesssim 1.$$ 

Firstly, by Bernstein’s inequality we may estimate for any $k \in \{3, \ldots, N\}$, 

$$\max_{1 \leq l \leq N} \| \partial_t u^k \|_{B_p} = \max_{1 \leq l \leq N} \| \partial_t u^k \|_{B_p^\alpha} \lesssim \| \partial_t u^k \|_{L^\infty} \lesssim \sum_j 2^{j(1-2\alpha + \frac{2\alpha}{p})} \| \Delta_j u^k \|_{L^\infty} \lesssim \sum_j 2^{j(\frac{3}{2} + 1 - 2\alpha + \frac{2\alpha}{p})} \| \Delta_j u^k \|_{L^2} \lesssim \| u^k \|_{H^{\frac{3}{2} - 2\alpha + \frac{2\alpha}{p}}}$$ 

as $L^2 \subset L^\infty$, and thus 

$$\max_{1 \leq l \leq N} \int_0^T \sum_{k=3}^N \| \partial_t u^k \|_{B_p}^p \, dt \lesssim \int_0^T \sum_{k=3}^N \| u^k \|_{H^{\frac{3}{2} - 2\alpha + \frac{2\alpha}{p}}}^p \, dt. \tag{39}$$ 

Next, 

$$\int_0^T \| \nabla_h u^k \|_{B_p}^p \, dt \lesssim \int_0^T \| \nabla_h \Delta_h^{-1} \omega^3 \|_{B_p}^p \, dt + \sum_{k=3}^N \int_0^T \| \nabla_h (\nabla_h \Delta_h^{-1} \partial_h u^k) \|_{B_p}^p \, dt$$ 

by Lemma 1.3.

Now we compute for $M$ to be chosen subsequently 

$$\| a \|_{B_p} \leq \| a \|_{B_p^{\alpha}} \lesssim \sum_{j \leq M} 2^{j(\frac{3}{2} - \frac{2\alpha}{p} + \frac{2\alpha}{p})} \| \Delta_j a \|_{L^2} + \sum_{j > M} 2^{j(-2\alpha + \frac{2\alpha}{p} + \frac{2\alpha}{p})} \| \Delta_j a \|_{L^2} \leq 2^M \| a \|_{L^2} + 2^{M(\frac{3}{2} - \frac{2\alpha}{p} + \frac{2\alpha}{p})} \| a \|_{L^2}$$ 

where we used that $L^2 \subset L^\infty$, Bernstein’s inequality, that $L^2 \subset L^\infty$, that 

$$-3\alpha + \frac{2\alpha}{p} + \frac{N}{2} < 0, \quad -2\alpha + \frac{2\alpha}{p} + \frac{N}{2} > 0$$
and Lemma 2.2. Choosing $M$ so that $2^M = \left( \frac{\|\Lambda^\alpha a\|_{L^2}}{\|a\|_{L^2}} \right)^{\frac{1}{\alpha}}$ in (41) leads to

$$\|a\|_{B^p} \lesssim \|\Lambda^\alpha a\|_{L^2}^{\frac{2+\frac{\alpha}{p}}{\alpha}} \|a\|_{L^2}^{\frac{3-\frac{2}{p}}{\alpha}}$$  \hspace{1cm} (42)

Thus, applying (42) in (40), we deduce

$$\int_0^T \|\nabla h(\nabla h \Delta_h^{-1} \omega^3)\|_{B^p}^p d\tau \lesssim \int_0^T \|\omega^3\|_{L^2}^{p(\frac{3-\frac{2}{p}}{\alpha} - \frac{N}{2\alpha})} \|\Lambda^\alpha \omega^3\|_{L^2}^{p(-2+\frac{\alpha}{p} + \frac{N}{2\alpha})} d\tau \lesssim 1.$$  \hspace{1cm} (43)

On the other hand,

$$\sum_{m=3}^N \int_0^T \|\partial_m u^h\|_{B^p}^p d\tau \lesssim \sum_{m=3}^N \int_0^T \sup_j 2^{j\left(\frac{3-\frac{2}{p}}{\alpha} - \frac{N}{2\alpha}\right)} \|\Delta_j \nabla_h \nabla h \Delta_h^{-1} \partial_k u^k\|_{L^\infty}^p d\tau \lesssim \sum_{m=3}^N \int_0^T \|u^k\|_{H^{\frac{3}{2} - 2\alpha + \frac{N}{2\alpha}}}^p d\tau \lesssim 1$$  \hspace{1cm} (44)

by Bernstein’s inequality and that

$$\|\nabla h \nabla h \nabla h^{-1} f\|_{L^2} = \left\|\nabla_h \nabla h \nabla h^{-1} f\right\|_{L^2} \lesssim \left\|f\right\|_{L^2} \approx \left\|f\right\|_{L^2}.$$  

due to continuity of Riesz transform in $\mathbb{R}^2$. In sum of (43) and (44) applied to (42), we obtain

$$\int_0^T \|\nabla h u^h\|_{B^p}^p d\tau \lesssim 1.$$  \hspace{1cm} (45)

Finally, we work on

$$\sum_{m=3}^N \int_0^T \|\partial_m u^h\|_{B^p}^2 d\tau \lesssim \left\|\partial_m \nabla_h \Delta_h^{-1} \omega^3\right\|_{B^p}^2 + \sum_{m=3}^N \left\|\partial_m \nabla h \Delta_h^{-1} \left(\sum_{l=3}^N \partial_l u^l\right)\right\|_{B^p}^2 d\tau$$  \hspace{1cm} (46)

by Lemma 1.3. This is in some sense the most difficult term because $\partial_m \nabla_h \Delta_h^{-1}$, $\partial_m \nabla h \Delta_h^{-1}$, $m = 3, \ldots, N$, are no longer just horizontal Riesz transforms. We may
estimate for some $\delta \in \mathbb{Z}_+$,

$$
\sum_{m=3}^{N} \int_0^T \left\| \partial_m \nabla_h^j \Delta_h^{-1} \omega^3 \right\|_{L^2}^2 \, dt
$$

(47)

and that

$$
\sum_{m=3}^{N} \int_0^T \left( \sup_j 2^{j(-\alpha)} \sum_{k \leq j+\delta} \sum_{l \leq j+\delta} 2^{(N-2)} \right) \left( \sum_{i,m=3}^{N} \left\| \Delta_j \Delta_h^k \Delta_l^i \varphi \partial_m \nabla_h \Delta_h^{-1} \omega^3 \right\|_{L^2} \right)^2 \, dt
$$

where we used Bernstein’s inequality and Plancherel theorem. We now continue this bound by

$$
\sum_{m=3}^{N} \int_0^T \left( \sup_j 2^{j(-\alpha)} \sum_{k \leq j+\delta} \sum_{l \leq j+\delta} 2^{(N-2)} \right) \left( \sum_{i,m=3}^{N} \left\| \Delta_j \Delta_h^k \Delta_l^i \varphi \partial_m \nabla_h \Delta_h^{-1} \omega^3 \right\|_{L^2} \right)^2 \, dt
$$

and that

$$
\sum_{m=3}^{N} \int_0^T \left( \sup_j 2^{j(-\alpha)} \sum_{k \leq j+\delta} \sum_{l \leq j+\delta} 2^{(N-2)} \right) \left( \sum_{i,m=3}^{N} \left\| \Delta_j \Delta_h^k \Delta_l^i \varphi \partial_m \nabla_h \Delta_h^{-1} \omega^3 \right\|_{L^2} \right)^2 \, dt
$$

(48)

by Hölder’s inequality, Lemma 2.2 and that

$$
\left\| \left( \Delta_l^i f \right) \right\|_{L^2} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \Delta_l^i f \right|^2 \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^2} \sum_{l} \left\| \Delta_l^i f \right\|_{L^2}^2 \, dx_h \right)^{\frac{1}{2}} = \left\| f \right\|_{L^2} \, dx_h
$$

Plancherel theorem and the uniform bound of $\Delta_j$ in $L^p(\mathbb{R}^N), p \in [1, \infty]$. On the very last term, it becomes crucial here that we have $H^{-\epsilon, \epsilon}, \epsilon > 0$. Firstly, we estimate for some $\delta \in \mathbb{N}$

$$
\sum_{i,m=3}^{N} \int_0^T \left\| \partial_m \nabla_h \Delta_h^{-1} \partial_i u^i \right\|_{L^2}^2 \, dt
$$

(49)
and continue by

\[
\sum_{i=3}^{N} \int_{0}^{T} \sup_{j} 2^{j(-\alpha)} \sum_{k,l \in [j-\delta,j+\delta]} \| \hat{\Delta}_j \hat{\Delta}_j^{\rho} \nabla \nabla \Delta^2 \hat{\Delta}_j^{\mu} \|_{L^\infty} \| \partial \|_{L^2}^2 d\tau \tag{50}
\]

by Bernstein’s inequalities, Plancherel theorem, Hölder’s inequality and uniform bound of \( \Delta \) in \( L^2(\mathbb{R}^N) \). Now we use the fact that \( \epsilon \in (0, \frac{N-2}{2}) \), Lemma 2.2 to continue to bound from (50) to

\[
\sum_{i=3}^{N} \int_{0}^{T} |\sup_{j} 2^{j(1-\frac{N}{2})} \sum_{k,l \in [j-\delta,j+\delta]} 2^{2k \epsilon 2 \lambda_2(\frac{N-2}{2} - \epsilon)} \| \hat{\Delta}_j \hat{\Delta}_j^{\rho} \nabla \nabla \Delta^2 \hat{\Delta}_j^{\mu} \|_{L^\infty} \| \partial \|_{L^2}^2 d\tau |^2 \tag{51}
\]

By (49), (50), (51), we conclude \( \sum_{i,m=3}^{N} \int_{0}^{T} \| \partial_i \Delta \|_{L^2}^2 d\tau \lesssim 1 \) and hence together with (47), (48), we obtain

\[
\sum_{m=3}^{N} \int_{0}^{T} \| \partial_m u \|_{L^2}^2 d\tau \lesssim 1.
\]

This, along with (39) and (45) concludes the proof of Theorem 1.1.

5. Appendix 1

The purpose of this Appendix is to announce some estimates that we were able to prove, and sketch their proofs briefly, in hope that they will help eliminate the conditions in (9) to just \( \sum_{k=3}^{N} \int_{0}^{T} \| u_k \|_{H^3-2a+\frac{2a}{p}}^p d\tau \) in future, in particular in the four-dimensional case as we believe this case to be the easiest among any other higher dimensions beyond three.

5.1. Several estimates. The following extensions of Lemma 4.2, Lemma 4.3 \[10\] may be proven.
Proposition 5.1. Let $s \in \mathbb{R}^+, p, q \in [1, \infty], p \geq q$. Then
\[ \|f\|_{L^p_k((B_{p,q}^{\alpha})_{s})} \lesssim \|f\|_{B_{p,q}^{\alpha}}. \] (52)

Proof. Firstly for any $l \in \mathbb{Z}$, there exists $N_0 \in \mathbb{Z}$ such that
\[ 2^{|s|} \|\hat{\Delta}^s f\|_{L^p} \lesssim 2^{|s|} \sum_{j \in \mathbb{Z} : l-j \leq N_0} 2^{|l-j|} 2^{sj} \|\hat{\Delta}^j f\|_{L^p} \] (53)
and therefore,
\[ \|(2^{|s|} \|\hat{\Delta}^s f\|_{L^p})l\|_{l^q(\mathbb{Z})} \lesssim \left( \sum_{j \in \mathbb{Z} : l-j \leq N_0} 2^{|l-j|} 2^{sj} \|\hat{\Delta}^j f\|_{L^p} \right)_{l} \lesssim \|f\|_{B_{p,q}^{\alpha}} \] (54)
by (53), Young’s inequality for convolution and that $s > 0$ by hypothesis. Hence,
\[ \|f\|_{L^p_k((B_{p,q}^{\alpha})_{s})} \lesssim \left( \|(2^{|s|} \|\hat{\Delta}^s f\|_{L^p})l\|_{l^q(\mathbb{Z})} \right)_{l^q(\mathbb{Z})} \lesssim \|f\|_{B_{p,q}^{\alpha}} \] (55)
by Minkowski’s inequality for integrals and (54). This completes the proof of Proposition 5.1.

Proposition 5.2. $\forall s > 0, \beta \in (0, s), \forall p, q \in [1, \infty],$
\[ \|f\|_{(B_{p,q}^{\alpha-\beta})_{s}} \lesssim \|f\|_{B_{p,q}^{\alpha}}. \] (56)

Proof. We let $V_k \triangleq \sum_{l \in \mathbb{Z}} 2^{|s|} \|\hat{\Delta}^l f\|_{L^p}$. We see that if we could show $V_k \lesssim c_k 2^{-k(s-\beta)} \|f\|_{B_{p,q}^{\alpha}}$ for some $(c_k)_k \in l^q(\mathbb{Z})$, then it would imply
\[ \sum_{l \in \mathbb{Z}} 2^{k(s-\beta)} 2^{|s|} \|\hat{\Delta}^l f\|_{L^p} \lesssim c_k \|f\|_{B_{p,q}^{\alpha}} \] (57)
and hence
\[ \|f\|_{(B_{p,q}^{\alpha-\beta})_{s}} = \|(2^{k(s-\beta)} \sum_{l \in \mathbb{Z}} 2^{|s|} \|\hat{\Delta}^l f\|_{L^p})_{l} \|_{l^q(\mathbb{Z})} \lesssim \|f\|_{B_{p,q}^{\alpha}} \] (58)
by (57) so that the proof may be complete. Thus, we aim to show $V_k \lesssim c_k 2^{-k(s-\beta)} \|f\|_{B_{p,q}^{\alpha}}$ for some $(c_k)_k \in l^q(\mathbb{Z})$. Firstly, $\hat{\Delta}^s f$ is uniformly bounded operators on $L^p(\mathbb{R}^4), p \in [1, \infty]$. Thus, for any fixed $k \in \mathbb{Z},$
\[ 2^{k(s-\beta)} V_k \lesssim 2^{k(s-\beta)} \sum_{l \in \mathbb{Z} : l \leq k} 2^{|l|} \|\hat{\Delta}^l f\|_{L^p} + 2^{k(s-\beta)} \sum_{l \in \mathbb{Z} : l > k} 2^{|l|} \|\hat{\Delta}^l f\|_{L^p} \lesssim 2^{k(s-\beta)} \|\hat{\Delta}^l f\|_{L^p} \] (59)
\[ \lesssim \sum_{j \in \mathbb{Z} : j \leq k-N_1} 2^{k(s-\beta)} \|\hat{\Delta}^j f\|_{L^p} + 2^{k(s-\beta)} \sum_{j \in \mathbb{Z} : j > k-N_0} 2^{-j(s-\beta)} 2^{|j|} \|\hat{\Delta}^j f\|_{L^p} \lesssim \sum_{j \in \mathbb{Z} : j \geq k-N_1} 2^{k(s-\beta)} 2^{|j|} \|\hat{\Delta}^j f\|_{L^p} + \sum_{j \in \mathbb{Z} : j \geq k-N_0} 2^{k(s-\beta)} 2^{|j|} \|\hat{\Delta}^j f\|_{L^p} \approx c_k \|f\|_{B_{p,q}^{\alpha}} \]}

$\forall q$ where we used Lemma 2.2, Littlewood-Paley decomposition and the hypothesis that $s > \beta$. □
The following is a kind of an extension of Proposition 4.1 [10].

**Proposition 5.3.** Let \( f = (f^1, f^2, f^3, f^4) \in C_0^\infty(\mathbb{R}^4) \) satisfy \( \nabla \cdot f = 0, g = \partial_t f^2 - \partial_2 f^1 \), and \( \epsilon > 0 \). Furthermore, suppose

\[ s_1 > 1, s_2 \in (\epsilon, s_1 - 1), s_1 + s_2 < 1 + \alpha. \]

Then

\[
\|f^h\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \lesssim \|g\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} + \sum_{m=3}^4 \|\partial_m f^m\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v}. \tag{60}
\]

**Remark 5.1.** In contrast to Proposition 4.1 [10], we found it difficult to let \( s_1 = 1 \); we explain its reason in the proof.

**Proof.** Firstly,

\[
\|f^h\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \lesssim \|g\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} + \sum_{m=3}^4 \|\partial_m f^m\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \tag{61}
\]

by Lemma 1.3. Now we work on

\[
\|g\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \lesssim \|g\|_{\dot{B}^{s_1+1}_{2,1}} \lesssim \sum_{j=1}^\infty 2^{j(s_1+s_2-1)}\|\hat{\Delta}_j g\|_{L^2} + \sum_{j>M} 2^{j(s_1+s_2-1)}\|\hat{\Delta}_j g\|_{L^2} \lesssim 2^{M(s_1+s_2-1)}\|g\|_{L^2} + 2^{M(s_1+s_2-1-\alpha)}\|\Lambda^\alpha g\|_{L^2} \tag{62}
\]

for \( M > 0 \) to be determined subsequently, by Proposition 5.2, Hölder’s inequality, Bernstein’s inequality and the fact that \( s_1 + s_2 - 1 > 0, s_1 + s_2 - 1 - \alpha < 0 \).

We highlight here that the issue when \( s_1 = 1 \) is that we cannot apply Proposition 5.2 because \( s_1 - 1 = 0 \neq 0 \). In [10] the authors get away with this issue here because they can take Bernstein’s inequality to go down from \((\dot{B}_{2,1}^s)\) to \((\dot{B}_{q,1}^\frac{n}{2}) = (\dot{B}_{\frac{n}{2},1}^\frac{n}{2})\) which is not an option in our case because \( \dot{W}^{1,p}(\mathbb{R}^4), p < 2 \) is not enough regularity for initial data to be even locally well-posed. However, this estimate at \( s_1 = 1 \) seems necessary in order to obtain the estimates on \( \|\omega^3\|_{L^2} \).

Now we choose \( M \) such that \( 2^M = \left(\frac{\|\Lambda^\alpha g\|_{L^2}}{\|g\|_{L^2}}\right)^\frac{1}{\alpha} \) so that

\[
\|g\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \lesssim \|g\|_{L^2}^{1-\frac{s_1+s_2-1}{\alpha}}\|\Lambda^\alpha g\|_{L^2}^{\frac{s_1+s_2-1}{\alpha}}. \tag{63}
\]

Next, we write

\[
\|\partial_m f^m\|_{(\dot{B}^{s_1+1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \tag{64}
\]

\[
= \sum_{k,l: k \leq l} 2^{k(s_1-1)}2^{ls_2}\|\hat{\Delta}_k \hat{\Delta}_l \partial_m f^m\|_{L^2} + \sum_{k,l: k > l} 2^{k(s_1-1)}2^{ls_2}\|\hat{\Delta}_k \hat{\Delta}_l \partial_m f^m\|_{L^2} \triangleq H_L(\partial_m f^m) + V_L(\partial_m f^m).
\]
We estimate for $M > 0$ to be determined subsequently

$$H_L(\partial_m f^m) \lesssim \sum_{k,l:k\leq M} 2^{k(s_1-1+\epsilon)} 2^{l(s_2-\epsilon)} \|\Delta_k h \Delta_l^\epsilon \partial_m f^m\|_2$$

(65)

$$+ \sum_{k,l:k/l > M} 2^{k(s_1-1+\epsilon)} 2^{l(s_2-\epsilon)} \|\Delta_k h \Delta_l^\epsilon \Lambda^\alpha \partial_m f^m\|_2.$$  

by Bernstein’s inequality and Plancherel theorem. Now we use the fact that $l^2 \subset l^\infty$, that $s_1 - 1 + \epsilon > 0$, Lemma 2.2 to continue to bound by

$$H_L(\partial_m f^m) \lesssim \|\partial_m f^m\|_{H^{-\epsilon,\epsilon}} 2^{M(s_1+s_2-1)} + \|\Lambda^\alpha \partial_m f^m\|_{H^{-\epsilon,\epsilon}} 2^{M(s_1+s_2-1-\alpha)}.$$  

(66)

Now we choose $M$ such that that $2^M = \left(\frac{\|\Lambda^\alpha \partial_m f^m\|_{H^{-\epsilon,\epsilon}}}{\|\partial_m f^m\|_{H^{-\epsilon,\epsilon}}}\right)^{\frac{1}{\alpha}}$ and hence

$$H_L(\partial_m f^m) \lesssim \|\partial_m f^m\|_{H^{-\epsilon,\epsilon}}^{\frac{1-s_1+s_2-1}{\alpha}} \|\Lambda^\alpha \partial_m f^m\|_{H^{-\epsilon,\epsilon}}^{\frac{s_1+s_2-1}{\alpha}}.$$  

(67)

Next,

$$V_L(\partial_m f^m) \lesssim \sum_{k,l:k/l \leq M} 2^{k(s_1-1+\epsilon)} 2^{l(s_2-\epsilon)} \|\Delta_k h \Delta_l^\epsilon \partial_m f^m\|_2$$

(68)

$$+ \sum_{k,l:k/l > M} 2^{k(s_1-1+\epsilon)} 2^{l(s_2-\epsilon)} \|\Delta_k h \Delta_l^\epsilon \Lambda^\alpha \partial_m f^m\|_2$$

by Bernstein’s inequality and Plancherel theorem. Now again we use the fact that $l^2 \subset l^\infty$, that $s_2 > \epsilon$, Lemma 2.2 to continue to bound by

$$V_L(\partial_m f^m) \lesssim \|\partial_m f^m\|_{H^{-\epsilon,\epsilon}} 2^{M(s_1+s_2-1)} + \|\Lambda^\alpha \partial_m f^m\|_{H^{-\epsilon,\epsilon}} 2^{M(s_1+s_2-1-\alpha)}.$$  

(69)

Now we choose $M$ such that $2^M = \left(\frac{\|\Lambda^\alpha \partial_m f^m\|_{H^{-\epsilon,\epsilon}}}{\|\partial_m f^m\|_{H^{-\epsilon,\epsilon}}}\right)^{\frac{1}{\alpha}}$, so that

$$V_L(\partial_m f^m) \lesssim \|\partial_m f^m\|_{H^{-\epsilon,\epsilon}}^{\frac{1-s_1+s_2-1}{\alpha}} \|\Lambda^\alpha \partial_m f^m\|_{H^{-\epsilon,\epsilon}}^{\frac{s_1+s_2-1}{\alpha}}.$$  

(70)

Therefore, we have shown by (61), (63), (64), (67), (70),

$$\|f^h\|_{(B^{s_1}_{2,1})_h(B^{s_2}_{2,1})_e} \lesssim \|g\|_2^{1-s_1+s_2-1} \|\Lambda^\alpha g\|_{L^2}^{s_1+s_2-1} + \sum_{m=3}^4 \|\partial_m f^m\|_{H^{-\epsilon,\epsilon}}^{1-s_1+s_2-1} \|\Lambda^\alpha \partial_m f^m\|_{H^{-\epsilon,\epsilon}}^{s_1+s_2-1}.$$  

(71)

This completes the proof of Proposition 5.3. □

We may also extend the inequalities (94), (95) of [37] to the four-dimensional case as follows:

**Proposition 5.4.** Let $N = 4$. For $s_1 \leq 1$, $s_2 \leq 1$, $s_2 + \frac{2\alpha}{p} - \theta > 0$, $s_1 + 2 - 2\alpha + \theta > 0$, $\theta \in \left(\frac{2\alpha}{p} - 1, 2\alpha - 1\right)$, it holds that for $f, g \in C^\infty_0(\mathbb{R}^4)$,

$$\|fg\|_{H^{s_1+1-2\alpha+s_2+\frac{2\alpha}{p}-1-\theta}} \lesssim \|f\|_{(B^{s_1}_{2,1})_h(B^{s_2}_{2,1})_e} \|g\|_{H^{2-2\alpha+s_2+\frac{2\alpha}{p}-\theta}}.$$  

(72)

The proof is standard (see e.g. [37]) and hence we omit it here for brevity. We also record in relevance the product law in anisotropic spaces (cf. Lemma 4.5 [10]) for which its proof is well-known (see e.g. [19, 36, 37]):
Proposition 5.5. Let $N = 4$. For $q \geq 1$, $p_1 \geq p_2 \geq 1$, $\frac{1}{p_1} + \frac{1}{p_2} = 1$, $s_1 < \frac{2}{p_1}$, $s_2 < \frac{2}{p_2}$ (resp. $s_1 \leq \frac{2}{p_1}$, $s_2 \leq \frac{2}{p_2}$ if $q = 1$), $s_1 + s_2 > 0$, $\sigma_1 < \frac{2}{p_1}$, $\sigma_2 < \frac{2}{p_2}$ (resp. $\sigma_1 \leq \frac{2}{p_1}$, $\sigma_2 \leq \frac{2}{p_2}$ if $q = 1$), $\sigma_1 + \sigma_2 > 0$, if $f, g \in C_0^\infty (\mathbb{R}^4)$, then

$$
\|fg\|_{(B^s_{p_1,q} \ast B^s_{p_2,q})_{h}(B^s_{p_1,q} \ast B^s_{p_2,q})_v} \lesssim \|f\|_{(B^{\sigma_1}_{p_1,q} \ast h)(B^{\sigma_1}_{p_1,q} \ast h)_v} \|g\|_{(B^{\sigma_2}_{p_2,q} \ast h)(B^{\sigma_2}_{p_2,q} \ast h)_v}.
$$

(73)

6. Acknowledgment

The author expresses gratitude to Dr. Vincent R. Martinez and Dr. Senjo Shimizu for fruitful discussion the editor and the referees for valuable review and comments that improved the manuscript greatly.

References


