GLOBAL REGULARITY OF N-DIMENSIONAL GENERALIZED MHD SYSTEM WITH ANISOTROPIC DISSIPATION AND DIFFUSION

KAZUO YAMAZAKI
Department of Mathematics, Washington State University
Pullman, WA 99164-3113

Abstract. Motivated by the anisotropic Navier-Stokes equations that has much applications in studying fluids in thin domains, in particular meteorology and oceanography, we study the magnetohydrodynamics system with generalized dissipation and diffusion, considering different exponents of the fractional Laplacians with logarithmic worsening applied to different directions and components of the solution vector fields. The results indicate that it is possible to regularize the flows by anisotropic dissipation, some components in various directions allowed to be below the critical exponents in the expense of others being above.

Keywords: Navier-Stokes equations, magnetohydrodynamics system, global regularity, fractional Laplacian, Besov spaces

1. Introduction and Statement of Theorems

We study the $N$-dimensional magnetohydrodynamics (MHD) system $N \geq 2$ defined as follows:

$$
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi &= \nu \Delta u, \\
\frac{\partial b}{\partial t} + (u \cdot \nabla)b - (b \cdot \nabla)u &= \eta \Delta b, \\
\nabla \cdot u &= \nabla \cdot b = 0, \quad (u, b)(x, 0) = (u_0, b_0)(x),
\end{align*}
$$

where $u : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}^N, b : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}^N, \pi : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}$ are the velocity, magnetic and pressure fields respectively and the parameters $\nu, \eta \geq 0$ represent the kinematic viscosity and diffusivity constants respectively. The system describes the motion of electrically conducting fluids and plays a fundamental role in applied sciences such as astrophysics, geophysics and plasma physics. Let us denote hereafter $\partial_t = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial x_i}$.

In both cases $N = 2, 3, \nu, \eta > 0$, the system possesses at least one global $L^2$-weak solution pair for any initial data pair $(u_0, b_0) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$; moreover, in case $N = 2$, the global existence of strong solution has been shown (cf. [18]). However, the global regularity issue of the solution pair in the case $N \geq 3$ remains open. In fact, because the system at $b \equiv 0$ is reduced to the Navier-Stokes equations (NSE), such an issue seems to be extremely challenging. For example, in case $N = 2$ while the inviscid NSE, the Euler equations, admits a global regularity result (e.g. [13]), it remains unknown whether the same result holds for the MHD system.

2000MSC : 35B65, 35Q35, 35Q86.

The author expresses gratitude to Professor Jiahong Wu and Professor David Ullrich for their teaching and referees for valuable comments and suggestions for improvement of the manuscript.
To suggest a new approach, the author in [25] studied the generalized MHD system; that is,
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi + \nu \lambda^{2\alpha} u &= 0, \\
\frac{\partial b}{\partial t} + (u \cdot \nabla)b - (b \cdot \nabla)u + \eta \lambda^{2\beta} b &= 0,
\end{align*}
where $\lambda = (-\Delta)^{\frac{1}{2}}$ defined through Fourier transform as follows:
\[\mathcal{F}(\lambda^{2\gamma} f)(\xi) = |\xi|^{2\gamma} \mathcal{F}(f)(\xi), \quad \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^N} f(x)e^{-ix \cdot \xi} dx.\]
We note that the classical MHD system (1) is the special case of (2), at $\alpha = \beta = 1$. After such a generalization, the author in [25] showed that in case $\nu, \eta > 0$ and $\alpha \geq \frac{1}{2} + \frac{N}{4}$, $\beta \geq \frac{1}{2} + \frac{N}{4}$, $(u, b)$ remains smooth for all time (cf. [35]). This lower bound is related to the fact that, for simplicity, taking $\alpha = \beta = \gamma$, $\lambda \in \mathbb{R}^+$, if $(u(x, t), b(x, t))$ solves (2), then so does $(u_\lambda(x, t), b_\lambda(x, t))$ where
\[u_\lambda(x, t) = \lambda^{2\gamma-1}u(\lambda x, \lambda^2 t), \quad b_\lambda(x, t) = \lambda^{2\gamma-1}b(\lambda x, \lambda^2 t),\]
and precisely when $\alpha = \beta = \frac{1}{2} + \frac{N}{4}$, we have $\|u_\lambda(\cdot, t)\|_{L^4(\mathbb{R}^N)} = \|u(\cdot, \lambda^2 t)\|_{L^4(\mathbb{R}^N)}$, $\|b_\lambda(\cdot, t)\|_{L^4(\mathbb{R}^N)} = \|b(\cdot, \lambda^2 t)\|_{L^4(\mathbb{R}^N)}$. Reducing this lower bound furthermore below $\frac{1}{2} + \frac{N}{4}$ has become an extremely difficult problem.

Motivated by the recent results in [19, 20] on the wave equations and the NSE respectively, the author in [26] improved his own result logarithmically. In short, he generalized the fractional Laplacians furthermore and considered
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi + \nu \mathcal{L}^2 u &= 0, \\
\frac{\partial b}{\partial t} + (u \cdot \nabla)b - (b \cdot \nabla)u + \eta \mathcal{M}^2 b &= 0, \\
\nabla \cdot u &= \nabla \cdot b = 0, \quad (u, b)(x, 0) = (u_0, b_0)(x),
\end{align*}
where $\mathcal{L}$ and $\mathcal{M}$ are defined by
\begin{align*}
\mathcal{F}(\mathcal{L}f)(\xi) &= l(\xi) \mathcal{F}(f)(\xi), \quad l(\xi) \geq \frac{|\xi|^\alpha}{\sigma(\xi)}, \\
\mathcal{F}(\mathcal{M}f)(\xi) &= m(\xi) \mathcal{F}(f)(\xi), \quad m(\xi) \geq \frac{|\xi|^\beta}{\pi(\xi)},
\end{align*}
with $\alpha \geq \frac{1}{2} + \frac{N}{4}, \beta > 0, \alpha + \beta \geq 1 + \frac{N}{2}$ and $g, h \geq 1$ are radially symmetric non-decreasing functions such that they satisfy a certain integral condition, and showed that this system has a unique global classical solution (see also [5, 8, 17, 23, 30]). Besides logarithmic improvement, these results showed that the exponents of the fractional Laplacians may be shifted toward that of dissipation rather than diffusion, as long as the sum continues to satisfy the lower bound of $\alpha + \beta \geq 1 + \frac{N}{2}$. We also note that e.g. in [10, 34] it was shown that the regularity criteria for this system may be shown to rely only on $u$, dropping conditions on $b$ completely.

In this paper we are concerned with the flexibility of the exponents in each direction and component of $u$ and $b$. In other words, denoting by $\mathcal{F}_i$ the one-dimensional Fourier transform in the $\xi_i$-direction, we define $\Lambda_i$ by
\[\mathcal{F}_i(\lambda^{2i\gamma} f)(\xi_i) = |\xi_i|^{2i\gamma} \mathcal{F}_i(f)(\xi_i), \quad i = 1, \ldots, N,
\]
so that e.g. $\lambda^{2\alpha} u$ with $\alpha = \frac{1}{2} + \frac{N}{4}$ may be understood as
\[
\mathcal{F}(\lambda^{1+\frac{N}{4}} u)(\xi) = |\xi|^{1+\frac{N}{4}} \mathcal{F}(u)(\xi) = (\xi_1^2 + \ldots + \xi_N^2)^{\frac{1}{2}+\frac{N}{4}} (\mathcal{F}(u_1)(\xi), \ldots, \mathcal{F}(u_N)(\xi))^T.
\]
Now we let \( \alpha_{i,j} \) be the exponent of the one-dimensional fractional Laplacian in the \( \cdot \)-th direction for the \( j \)-th component of \( u \); i.e.,

\[
\mathcal{F}_i(\Lambda_{i}^{2\alpha_{i,j}}u_j)(\xi_i) = |\xi_i|^{2\alpha_{i,j}}\mathcal{F}_i(u_j)(\xi_i),
\]

and similarly with \( \beta_{i,j} \) for \( b_j \). We consider

\[
L^2 u \triangleq \begin{pmatrix}
\sum_{i=1}^{N} L_{i,1}^2 u_1 \\
\sum_{i=1}^{N} L_{i,2}^2 u_2 \\
\vdots \\
\sum_{i=1}^{N} L_{i,N}^2 u_N
\end{pmatrix},
\]

\[
M^2 b \triangleq \begin{pmatrix}
\sum_{i=1}^{N} M_{i,1}^2 b_1 \\
\sum_{i=1}^{N} M_{i,2}^2 b_2 \\
\vdots \\
\sum_{i=1}^{N} M_{i,N}^2 b_N
\end{pmatrix},
\]

where

\[
\mathcal{F}_i(L_{i,j}u_j)(\xi_i) = l_{i,j}(\xi_i)\mathcal{F}_i(u_j)(\xi_i),
\]

\[
\mathcal{F}_i(M_{i,j}b_j)(\xi_i) = m_{i,j}(\xi_i)\mathcal{F}_i(b_j)(\xi_i),
\]

and investigate what choices of lower bounds we must require as in (4) in order to guarantee the global regularity of \((u, b)\). We present our results:

**Theorem 1.1.** Suppose \( N = 2, 3, 4 \) or 5 and

\[
\begin{align*}
\begin{cases}
\quad l_{i,j}(\xi_i) \geq |\xi_i|^\alpha_{i,j}, \\
\quad n_{i,j}(\xi_N) \geq \frac{|\xi_N|^{\alpha_{i,j}}}{g_{N,j}(\xi_N)}, \\
\quad \alpha_{i,j} \geq \frac{N}{2},
\end{cases}
\end{align*}
\]

for \( j = 2, \ldots, N, \)

\[
\begin{align*}
\begin{cases}
\quad m_{i,j}(\xi_i) \geq |\xi_i|^\beta_{i,j}, \\
\quad m_{i,N}(\xi_N) \geq |\xi_N|^\beta_{i,N},
\end{cases}
\end{align*}
\]

while

\[
\begin{align*}
\begin{cases}
\quad m_{i,j}(\xi_i) \geq |\xi_i|^\beta_{i,j}, \\
\quad m_{i,N}(\xi_N) \geq |\xi_N|^\beta_{i,N},
\end{cases}
\end{align*}
\]

for \( j = 2, \ldots, N, \)

where \( g_{N,j}, h_{N,j}, j = 2, \ldots, N \) are radially symmetric non-decreasing functions such that \( 1 \leq g_{N,j}(\tau), h_{N,j}(\tau) \leq c\sqrt{\ln(1 + \ln(\tau))}, \forall \tau \in [0, \infty) \). Then for every \((u_0, b_0) \in H^s(\mathbb{R}^N), s > 1 + \frac{N}{2}, \) \( s \geq 1 + \max_{i,j} \{2\alpha_{i,j}, 2\beta_{i,j}\} \), there exists a global classical solution to (3) with \( L, M \) defined by (5).

We immediately remark that the hypothesis on \( g_{N,j}, h_{N,j} \) imply

\[
\int_{\tau}^\infty \frac{d\tau}{\sum_{j=2}^{N} (g_{N,j}(\tau) + h_{N,j}(\tau)) \ln(\tau)} = \infty.
\]

**Theorem 1.2.** Suppose \( N \in \mathbb{N}, N \geq 6 \) and

\[
\begin{align*}
\begin{cases}
\quad l_{i,j}(\xi_i) \geq |\xi_i|^\alpha_{i,j}, \\
\quad l_{i,j}(\xi_N) \geq |\xi_N|^\alpha_{i,N},
\end{cases}
\end{align*}
\]

for \( j = 2, \ldots, N, \)

\[
\begin{align*}
\begin{cases}
\quad m_{i,j}(\xi_i) \geq |\xi_i|^\beta_{i,j}, \\
\quad m_{i,N}(\xi_N) \geq |\xi_N|^\beta_{i,N},
\end{cases}
\end{align*}
\]

where
exists a global classical solution to (3) with
\[ m_{i,j}(\xi_i) \geq |\xi_i|^{\beta_{i,j}}, \quad \beta_{i,j} \geq \max\{ \frac{2M}{N_j}, \frac{N_j - 1}{2} \}, \quad i = 1, \ldots, N - 1, \]
\[ m_{N,j}(\xi_N) \geq |\xi_N|^{\beta_{N,j}}, \quad \beta_{N,j} \geq N \cdot \frac{1}{2}, \]
for \( j = 2, \ldots, N, \)
\[ m_{i,1}(\xi_i) \geq |\xi_i|^{\beta_{i,1}}, \quad \beta_{i,1} \geq N - 1, \quad i = 1, \ldots, N - 1, \]
\[ m_{N,1}(\xi_N) \geq |\xi_N|^{\beta_{N,1}}, \quad \beta_{N,1} \geq N \cdot \frac{1}{2}. \]

Then for every \((u_0, b_0) \in H^s(\mathbb{R}^N),\) \(s > 1 + \frac{N}{2},\) \(s \geq 1 + \max_{i,j}\{2\alpha_{i,j}, 2\beta_{i,j}\},\) there exists a global classical solution to (3) with \(\mathcal{L}, \mathcal{M}\) defined by (5).

**Remark 1.1.**

(1) Let us emphasize that in [26] as we explained in (4), the condition of \(\alpha \geq \frac{1}{2} + \frac{N}{4}\) implies the exponent \(\alpha\) on each component of \(u = (u_1, \ldots, u_N)\) and each direction \(x_1, \ldots, x_N\) and hence in sum at least \(N^2(\frac{1}{2} + \frac{N}{4})\). More importantly, this implies that in [26], no component of \(u\) is allowed to have the exponent less than \(\frac{1}{2} + \frac{N}{4}\) in any direction. The novelty of Theorems 1.1, 1.2 is that for some components, in some direction, such supercritical exponent is allowed, although in the expense of subcritical exponents in others.

For example, from Theorem 1.1 in case \(N = 2,\) we in particular showed global regularity under the dissipation and diffusion strengths of
\[
\mathcal{F}(\mathcal{L}^2 u)(\xi) = \left( \frac{|\xi_1|^2 \mathcal{F}(u_1)}{|\xi_1|^3 \mathcal{F}(u_2) + \frac{M_2}{\ln(e + \ln(e + |\xi_1|))} \mathcal{F}(u_2)} \right) \]
\[
\mathcal{F}(\mathcal{M}^2 b)(\xi) = \left( \frac{|\xi_1|^2 \mathcal{F}(b_1)}{|\xi_1|^3 \mathcal{F}(b_2) + \frac{M_2}{\ln(e + \ln(e + |\xi_1|))} \mathcal{F}(b_2)} \right).\]

We remark that in the case \(N = 2,\) very recently we have seen many interesting developments ([4, 9, 12, 21, 22, 28, 31, 32]). However, in all these cases, they require at least \(\Delta b\) in the diffusive term; i.e. \(\beta_{i,j} \geq 1 \forall i,j = 1,2\) without any division by a logarithmic function while above we have \(\beta_{1,1} = \beta_{2,1} = \frac{3}{4}.\) We also remark that in [3], the authors obtained global regularity results with mixed dissipation and diffusion; for example, the case \(\alpha_{1,1} = \alpha_{1,2} = \alpha_{2,1} = \alpha_{2,2} = \alpha_{1,1} = \beta_{1,2} = 0\) without any division by a logarithmic function. Finally, the authors in [2] studied sufficient conditions for the global regularity in case \(N = 2\) with vertical dissipation and vertical diffusion; i.e. \(\alpha_{2,1} = \alpha_{2,2} = \beta_{2,1} = \beta_{2,2} = 1, \alpha_{1,1} = \alpha_{1,2} = \beta_{1,1} = \beta_{1,2} = 0.\) Our result is not covered by any of these cases.

(2) The proof was inspired by the work of many others: [15, 29, 33]. While this manuscript was in preparation, the work of [24] appeared from which we were also partially inspired. Initial regularity may be improved or generalized by various methods; for simplicity we chose to state above which may be proven using standard mollifiers method (cf. [16]).

(3) Both works in [24, 33] rely heavily on Lemma 2.3 of [15] which does not seem to be easily extended to higher dimension due to the lack of cancellations. Moreover, a decomposition of non-linear terms separating only one velocity component such as Lemma 2.3 of [15] does not exist in the case of the MHD system due to the addition of three more non-linear terms mixed...
with $b$ even in case $N = 3$ (see e.g. [11, 27]). In Proposition 3.1, we find a decomposition that allows us to overcome this difficulty.

(4) In relevance, we mention that anisotropic NSE is the classical NSE with its dissipation operator $\nu \Delta$ replaced by $\nu_b \Delta_h + \nu_c \partial_i^2$ where $\Delta_h = \partial_1^2 + \partial_2^2$ and the viscosity coefficient $\nu$ separated to the horizontal and vertical parts. It has much applications in studying fluids in thin domains (cf. [7]) and analogously anisotropic MHD is also said to have potential applications e.g. in thin layers of sun.

In the Preliminaries section, we set up notations and state key lemmas. Thereafter, we prove Theorem 1.1, leaving the proof of Theorem 1.2 in the Appendix as it is similar and also the proofs of Lemmas 2.2, 2.3 and 2.4.

2. Preliminaries

We write $A \lesssim_{a,b} B$ when there exists a constant $c = c(a,b) \geq 0$ of no significant dependence on any other parameter such that $A \leq cB$. Similarly we write $A \approx B$ in case $A = cB$. Let us also write $(N-1)$-dimensional gradient vector

$$\nabla_{N-1} = (\partial_1, \partial_2, \ldots, \partial_{N-1}, 0).$$

For brevity we write $\int f = \int_{\mathbb{R}^N} f(x) \, dx$ and $\| f \|_{L^p} = \| f \|_{L^p(\mathbb{R}^N)}$. We now recall the notion of mixed $L^p$-space (cf. [1]) and denote

$$\| f \|_{L^p_{\xi_1} L^q_{\xi_2}} = \| f \|_{L^p_{\xi_1}(L^q_{\xi_2})} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f|^q \, dx_j \right)^{\frac{p}{q}} \, dx_i \right)^{\frac{1}{p}},$$

and emphasize that the order matters as due to Minkowski’s inequality of integrals,

$$\| f \|_{L^p_{\xi_1} L^q_{\xi_2}} \leq \| f \|_{L^p_{\xi_1} L^q_{\xi_2}} \quad \text{if } 1 \leq p \leq q \leq \infty.$$

Moreover, let us also denote by

$$\| f \|_{H^s_{\xi}}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^{s} |\mathcal{F}(f)(\xi)|^2 \, d\xi,$$

the one-dimensional inhomogeneous Sobolev space of order $\gamma$ in the $\xi_i$-direction.

Let us recall the notion of Besov spaces (cf. [6]). We denote by $\mathcal{S}(\mathbb{R}^N)$ the Schwartz class functions and $\mathcal{S}'(\mathbb{R}^N)$, its dual. We define

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int \phi(x) x^\delta \, dx = 0, |\delta| = 0, 1, 2, \ldots \right\}.$$

Its dual $\mathcal{S}'_0$ is given by $\mathcal{S}'_0 / \mathcal{S}'_0^\perp = \mathcal{S}' / \mathcal{P}$ where $\mathcal{P}$ is the space of polynomials. For $k \in \mathbb{Z}$ we define

$$A_k = \{ \xi \in \mathbb{R}^N : 2^{k-1} < |\xi| < 2^{k+1} \}.$$

It is well-known that there exists a sequence $\{ \Phi_k \} \subseteq \mathcal{S}(\mathbb{R}^N)$ such that

$$\operatorname{supp} \mathcal{F}(\Phi_k) \subset A_k, \mathcal{F}(\Phi_k)(\xi) = \mathcal{F}(\Phi_0)(2^{-k}\xi), \quad \sum_{k=\infty}^{\infty} \mathcal{F}(\Phi_k)(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^N \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Consequently, for any $f \in \mathcal{S}'_0$, we may set the Littlewood-Paley operators

$$\Delta_k f = \begin{cases} 0 & \text{if } k \leq -2, \\ \Psi \ast f & \text{if } k = -1, \\ \Phi_k \ast f & \text{if } k = 0, 1, 2, \ldots \end{cases}$$
Similarly, Lemma 2.4. For any \( \Psi \in C^\infty_0(\mathbb{R}^N) \) satisfies
\[
1 = \mathcal{F}(\Psi)(\xi) + \sum_{k=0}^{\infty} \mathcal{F}(\Phi_k)(\xi), \quad \Psi * f + \sum_{k=0}^{\infty} \Phi_k * f = f
\]
and define for \( s \in \mathbb{R}, p, q \in [1, \infty] \), the inhomogeneous Besov space
\[
B^s_{p, q} = \{ f \in S' : \| f \|_{B^s_{p, q}} < \infty \}, \quad \| f \|_{B^s_{p, q}} = \begin{cases} \| 2^{ks} \| \Delta_k f \|_{L^p} \|_{\ell^{q}} & \text{if } q < \infty, \\ \sup_{1 \leq k < \infty} 2^{ks} \| \Delta_k f \|_{L^p} & \text{if } q = \infty. \end{cases}
\]
In particular \( B^s_{2, 2} = H^s \). Finally, we state the Bernstein’s inequality:

**Lemma 2.1.** (cf. [6]) Let \( f \in L^p \) with \( 1 \leq p \leq q \leq \infty, \ 0 < r < R \). Then for all \( \delta \in \mathbb{Z}^+ \cup \{0\} \), and \( \lambda > 0 \), there exists a constant \( C_\delta > 0 \) such that
\[
\left\{ \begin{array}{ll}
\sup_{|\gamma| = \delta} \| \partial^\gamma f \|_{L^p} \leq C_\delta \lambda^{\delta + N(\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p} & \text{supp } \mathcal{F}(f) \subset \{ \xi : |\xi| \leq \lambda r \}, \\
\frac{\lambda^s}{c_{rs}} \| f \|_{L^p} \leq \sup_{|\gamma| = \delta} \| \partial^\gamma f \|_{L^p} \leq C_\delta \lambda^s \| f \|_{L^p} & \text{supp } \mathcal{F}(f) \subset \{ \xi : \lambda r \leq |\xi| \leq \lambda R \},
\end{array} \right.
\]
and if we replace derivative \( \partial^\gamma \) by the fractional derivative, the inequalities remain valid only with trivial modifications.

In order to obtain the logarithmically less dissipation and diffusion case we will need the following lemma, which is a slight variation of those in [24, 30, 33]:

**Lemma 2.2.** Suppose \( \mathcal{N}_i \) is defined by
\[
\mathcal{F}_i(\mathcal{N}_i \phi)(\xi_i) = n_i(\xi_i) \mathcal{F}_i(\phi)(\xi_i), \quad n_i(\xi_i) = \left| \frac{\xi_i}{G_i}(\xi_i) \right|^\gamma, \quad \gamma \geq 0, \quad \phi \in C^\infty_0(\mathbb{R})
\]
where \( G_i \geq 1 \) is a radially symmetric non-decreasing function. Then for any \( A \geq 0 \), there exists a constant \( c \geq 0 \) such that
\[
\| \phi \|_{L^\infty(\mathbb{R})} \leq c \left( G_i(A)^\sqrt{\ln(A)} \| \mathcal{N}_i \phi \|_{L^2(\mathbb{R})} + \frac{1}{\sqrt{A}} \| \phi \|_{H^1(\mathbb{R})} \right).
\]
Similarly,
\[
\| \phi \|_{L^\infty(\mathbb{R})} \leq c \left( \sqrt{\ln(A)} \| \phi \|_{H^\frac{1}{2}(\mathbb{R})} + \frac{1}{\sqrt{A}} \| \phi \|_{H^1(\mathbb{R})} \right).
\]

**Lemma 2.3.** Suppose \( \mathcal{N}_i \) is defined by
\[
\mathcal{F}_i(\mathcal{N}_i \phi)(\xi_i) = n_i(\xi_i) \mathcal{F}_i(\phi)(\xi_i), \quad n_i(\xi_i) = \left| \frac{\xi_i}{G_i}(\xi_i) \right|, \quad \gamma \geq \frac{3}{2}, \quad \phi \in C^\infty_0(\mathbb{R})
\]
where \( 1 \leq G_i(\tau) \leq c \sqrt{\ln(e + \ln(\tau))} \) is a radially symmetric non-decreasing function. Then for any \( A \geq 0 \), there exists a constant \( c \geq 0 \) such that
\[
\| \partial_i \phi \|_{L^\infty(\mathbb{R})} \leq c \left( G_i(A)^\sqrt{\ln(A)} \| \mathcal{N}_i \phi \|_{L^2(\mathbb{R})} + \frac{1}{\sqrt{A}} \| \partial_i \mathcal{N}_i \phi \|_{L^2(\mathbb{R})} \right).
\]

We also need the following lemmas:

**Lemma 2.4.** For any \( \phi \in C^\infty_0(\mathbb{R}^N) \), there exists a constant \( c \geq 0 \) such that
\[
\| \phi \|_{L^2_{-1}(\mathbb{R}^N)} \leq c \| \phi \|_{L^2_{-1}}^\frac{1}{2} \| A^{\alpha} \phi \|_{L^2_{-1}}^\frac{1}{2}. \]

**Lemma 2.5.** (cf. [14]) Let \( f, g \in C^\infty_0(\mathbb{R}^N), p \in (1, \infty), \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_3}, p_1, p_2, p_3 \in (1, \infty), s > 0 \). Then there exists a constant \( c \geq 0 \) such that
\[
\| A^s(fg) - f A^s g \|_{L^p} \leq c(\| \nabla f \|_{L^{p_1}} \| A^{s - 1} g \|_{L^{p_2}} + \| A^s f \|_{L^{p_3}} \| g \|_{L^{p_3}}).
\]
Proof. Through integration by parts and using divergence-free properties, we have

\[
l_{N,j}(\xi_N) = \frac{|\xi_N|^{\alpha_{N,j}}}{g_{N,j}(\xi_N)}, \quad m_{N,j}(\xi_N) = \frac{|\xi_N|^{\beta_{N,j}}}{h_{N,j}(\xi_N)}
\]

\[\forall j = 2, \ldots, N \text{ in (6) and (8) as the other case is easier. We first prove the following proposition, which may be of independent interest.}

**Proposition 3.1.** Suppose \((u, b)\) solves (3) in \(\mathbb{R}^N, N \in \mathbb{N}, N \geq 2\). Then there exists a universal constant \(c \geq 0\) such that

\[
\int (u \cdot \nabla) u \cdot \Delta u - (b \cdot \nabla)b \cdot \Delta b + (u \cdot \nabla)b \cdot \Delta b - (b \cdot \nabla)u \cdot \Delta b
\]

\[
\leq c \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left( |\partial_i u_j| + |\partial_N u_j| \right) (|\nabla u|^2 + |\nabla b|^2) + (|\partial_t b_j| + |\partial_N b_j|) |\nabla u| |\nabla b|.
\]

Proof. Through integration by parts and using divergence-free properties, we have

\[
\int (u \cdot \nabla) u \cdot \Delta u + (u \cdot \nabla)b \cdot \Delta b
\]

\[
= - \sum_{j,k=1}^{N} \int \partial_k u_1 \partial_1 u_j \partial_k u_j + \partial_k u_1 \partial_1 b_j \partial_k b_j - \sum_{i=2}^{N} \sum_{j,k=1}^{N} \int \partial_k u_i \partial_i u_j \partial_k u_j + \partial_k u_i \partial_i b_j \partial_k b_j
\]

\[
= - \sum_{k=1}^{N} \int \partial_k u_1 \partial_1 u_1 \partial_k u_1 + \partial_k u_1 \partial_1 b_1 \partial_k b_1 - \sum_{j=2}^{N} \sum_{k=1}^{N} \int \partial_k u_1 \partial_1 u_j \partial_k u_j + \partial_k u_1 \partial_1 b_j \partial_k b_j
\]

\[
- \sum_{i=2}^{N} \sum_{j,k=1}^{N} \int \partial_k u_i \partial_i u_j \partial_k u_j + \partial_k u_i \partial_i b_j \partial_k b_j
\]

\[
= \sum_{n=2}^{N} \sum_{k=1}^{N} \int \partial_k u_1 \partial_1 u_n \partial_k u_1 + \partial_k u_1 \partial_1 b_n \partial_k b_1 - \sum_{j=2}^{N} \sum_{k=1}^{N} \int \partial_k u_1 \partial_1 u_j \partial_k u_j + \partial_k u_1 \partial_1 b_j \partial_k b_j
\]

\[
- \sum_{i=2}^{N} \sum_{j,k=1}^{N} \int \partial_k u_i \partial_i u_j \partial_k u_j + \partial_k u_i \partial_i b_j \partial_k b_j
\]

\[
\leq \sum_{n=2}^{N} \int |\partial_n u_n| |\nabla u|^2 + |\partial_n b_n| |\nabla u| |\nabla b| + \sum_{j=2}^{N} \int |\partial_j u_j| |\nabla u|^2 + |\partial_j b_j| |\nabla u| |\nabla b|
\]

\[
+ \sum_{i=2}^{N} \sum_{k=1}^{N} \int |\partial_k u_i| (|\nabla u|^2 + |\nabla b|^2)
\]

\[
\leq \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left( |\partial_i u_j| + |\partial_N u_j| \right) (|\nabla u|^2 + |\nabla b|^2) + (|\partial_t b_j| + |\partial_N b_j|) |\nabla u| |\nabla b|.
\]
Similarly,
\[
\int (b \cdot \nabla) b \cdot \Delta u + (b \cdot \nabla) u \cdot \Delta b
= - \sum_{j,k=1}^{n} \int \partial_{j} b_{i} \partial_{j} \partial_{k} u_{j} + \partial_{k} b_{i} \partial_{i} u_{j} \partial_{k} b_{j} - \sum_{i=2}^{N} \sum_{j,k=1}^{n} \int \partial_{k} b_{i} \partial_{j} \partial_{k} u_{j} + \partial_{k} b_{i} \partial_{i} u_{j} \partial_{k} b_{j}
\]
\[
= - \sum_{j,k=1}^{n} \int \partial_{j} b_{i} \partial_{j} \partial_{k} u_{j} + \partial_{k} b_{i} \partial_{i} u_{j} \partial_{k} b_{j} - \sum_{i=2}^{N} \sum_{j,k=1}^{n} \int \partial_{k} b_{i} \partial_{j} \partial_{k} u_{j} + \partial_{k} b_{i} \partial_{i} u_{j} \partial_{k} b_{j}
\]
\[
\leq \sum_{j,k=1}^{n} \int |\partial_{j} b_{i}| |\nabla b||\nabla u| + |\partial_{n} u_{n}| |\nabla b|^{2} + \sum_{j=2}^{N} \int |\partial_{1} b_{j}| |\nabla b||\nabla u| + |\partial_{2} u_{j}| |\nabla b|^{2}
+ \sum_{i=2}^{N} \sum_{k=1}^{n} \int |\partial_{i} b_{k}| |\nabla u| |\nabla b|
\]
\[
\leq \sum_{i=1}^{N-1} \sum_{j=2}^{N} \int (|\partial_{i} u_{j}| + |\partial_{n} u_{j}|)(|\nabla u|^{2} + |\nabla b|^{2}) + (|\partial_{i} b_{j}| + |\partial_{n} b_{j}|)|\nabla u| |\nabla b|.
\]
This completes the proof of Proposition 3.1.

We take \(L^{2}\)-inner products of (3) with \(u\) and \(b\) respectively and integrate in time to obtain
\[
\frac{1}{2} \sup_{t \in [0, T]} (||u||_{L^{2}}^{2} + ||b||_{L^{2}}^{2}) (t) + \sum_{i,j=1}^{N} \int_{0}^{T} ||L_{i,j} u_{j}||_{L^{2}}^{2} + ||M_{i,j} b_{j}||_{L^{2}}^{2} d\tau \lesssim 1. \tag{15}
\]

Now let us denote with \(s\) of the initial regularity of \((u_{0}, b_{0})\),
\[
X(t) \triangleq (||\nabla u||_{L^{2}}^{2} + ||\nabla b||_{L^{2}}^{2})(t), \quad Z(t) \triangleq (||\Lambda^{s} u||_{L^{2}}^{2} + ||\Lambda^{s} b||_{L^{2}}^{2})(t), \tag{16}
\]
\[
L(t) \triangleq \sum_{i,j=1}^{N} ||L_{i,j} u_{j}||_{L^{2}}^{2} + ||M_{i,j} b_{j}||_{L^{2}}^{2}, \quad D(t) \triangleq \sum_{i,j=1}^{N} ||\nabla L_{i,j} u_{j}||_{L^{2}}^{2} + ||\nabla M_{i,j} b_{j}||_{L^{2}}^{2},
\]
\[
E(t) \triangleq \sum_{i,j=1}^{N} ||\Lambda^{s} L_{i,j} u_{j}||_{L^{2}}^{2} + ||\Lambda^{s} M_{i,j} b_{j}||_{L^{2}}^{2}.
\]

**Proposition 3.2.** Suppose \((u, b)\) solves (3) where \(\{L_{i,j}\}, \{M_{i,j}\}\) satisfies (6)-(9) in \(\mathbb{R}^{N} \times [0, T], N = 2, 3, 4\) or 5. Then
\[
\sup_{t \in [0, T]} X(t) + \int_{0}^{T} D(\tau) d\tau < \infty.
\]
Proof. Taking $L^2$-inner products of (3) with $-(\Delta u, -\Delta b)$ and using Proposition 3.1 we obtain
\[
\frac{1}{2} \partial_t X(t) + D(t) \lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \int [\partial_t u_j |(\nabla u)^2 + (\nabla b)^2| + |\partial_N u_j|(\nabla u)^2 + (\nabla b)^2]
\]
\[+ |\partial_b b_j| \nabla u |\nabla b| + |\partial_N b_j| \nabla u |\nabla b| \approx \sum_{k=1}^{N} I_k. \tag{17}
\]
We will frequently use the following elementary inequality:
\[(a + b)^{p} \leq 2^{p}(a^{p} + b^{p}), \quad \forall\ 0 \leq p < \infty, \quad a, b \geq 0. \tag{18}\]

Firstly,
\[I_1 \approx \sum_{i=1}^{N-1} \sum_{j=2}^{N} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} [\partial_t u_j |\nabla u|^2 dxN \ldots dx1 \tag{19}\]
\[\lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \int_{\mathbb{R}^{N-1}} [\nabla u_j |L_{N}^\infty \| \nabla u \|_{L_{2N}^1}^2 dx1 \ldots dx_{N-1} \]
\[\lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \int_{\mathbb{R}^{N-1}} \left( g_{N,j}^{1/N} (A) \sqrt{\ln(\lambda)} \|N_N \partial_t u_j \|_{L_{2N}^1} + \frac{1}{\sqrt{A}} \|\partial_t u_j \|_{H_{1}^1} \right) \times \|\nabla u \|_{L_{2N}^1}^2 dx1 \ldots dx_{N-1} \]
by H"{o}lder’s inequalities and Lemma 2.2 with $\phi = \partial_t u_j, G_i = g_{N,j}$ from (6) and $\gamma = \frac{1}{2\sigma_{N,j}}$. We further bound by
\[I_1 \lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left( g_{N,j}^{1/N} (A) \sqrt{\ln(\lambda)} \|\nabla N \partial_t u_j \|_{L_{2N}^1} + \frac{1}{\sqrt{A}} \|\partial_t u_j \|_{L_{2N}^1} + \|\partial_{N}^2 u_j \|_{L_{2N}^1} \right) \times \|\nabla u \|_{L_{2N}^1} \|\nabla_{N-1} \|_{L_{2N}^1}^{N-1} \|u\|_{L_{2N}^1}^2 \]
\[\lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left( g_{N,j}^{1/N} (A) \sqrt{\ln(\lambda)} \|\nabla N \partial_t u_j \|_{L_{2N}^1} + \frac{1}{\sqrt{A}} \|\partial_t u_j \|_{L_{2N}^1} + \|\partial_{N}^2 u_j \|_{L_{2N}^1} \right) \times \|\nabla u \|_{L_{2N}^1} \|\nabla_{N-1} \|_{L_{2N}^1}^{N-1} \|u\|_{L_{2N}^1}^2 \]
\[\leq \epsilon_1 \|\nabla \nabla_{N-1}\|_{L_{2N}^1}^{N-1} \|u\|_{L_{2N}^1}^2 \]
\[+ c \|\nabla \|_{L_{2N}^1}^2 \left( \sum_{i=1}^{N-1} \sum_{j=2}^{N} g_{N,j}^{1/N} (A) \ln(\lambda) \|N_N \partial_t u_j \|_{L_{2N}^1} + \frac{1}{\sqrt{A}} \|\partial_t u_j \|_{L_{2N}^1} + \|\partial_{N}^2 u_j \|_{L_{2N}^1} \right) \]
due to H"{o}lder’s inequality, Lemma 2.4, Young’s inequality and (18). Now
\[\|\nabla \nabla_{N-1}\|_{L_{2N}^1}^{N-1} \|u\|_{L_{2N}^1}^2 \lesssim \sum_{i=1}^{N-1} \sum_{j=1}^{N} \int |\xi|^2 |\xi_i|^{N-1} F(u_j)^2 \tag{20}\]
\[\lesssim \sum_{i=1}^{N-1} \sum_{j=1}^{N} \int |\xi|^2 (1 + |\xi_i|^{2\alpha_i,j}) F(u_j)^2 \lesssim X(t) + D(t) \]
where we used the Plancherel theorem, (18) and Young’s inequalities due to (6), (7). Next,

\[
\sum_{i=1}^{N-1} \sum_{j=2}^{N} \|\mathcal{N}_i \partial_i u_j\|_{L^2}^2 = \sum_{i=1}^{N-1} \sum_{j=2}^{N} \int \frac{|\xi_N||\xi_j|^2}{g_{N,i}^{||\xi_N||}}(\xi_N) \mathcal{F}(u_j)^2
\]

(21)

\[
\lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left( 1 + |\xi_i|^{2\alpha_{i,j}} + \frac{|\xi_N|^{2\alpha_{N,j}}}{g_{N,i}^{||\xi_N||}}(\xi_N) \right) \mathcal{F}(u_j)^2 \lesssim 1 + L(t)
\]

by Plancherel theorem, Young’s inequality, (6) and (15). Moreover, for any \(i = 1, \ldots, N-1, j = 2, \ldots, N,\)

\[
\|\partial^2_{N,i} u_j\|_{L^2}^2 \lesssim \int |\xi_i|^2 (1 + |\xi_i|^{2\alpha_{i,j}}) \mathcal{F}(u_j)^2 \lesssim \|\nabla u_j\|_{L^2}^2 + \|\nabla \Lambda_{i,j} u_j\|_{L^2}^2,
\]

where we used Plancherel theorem, Young’s inequality with (6). Therefore,

\[
\sum_{i=1}^{N-1} \sum_{j=2}^{N} \|\partial^2_{N,i} u_j\|_{L^2}^2 \lesssim X(t) + D(t).
\]

(22)

Hence, by using (20)-(22) in (19), we obtain

\[
I_1 \lesssim \varepsilon_1(X(t) + D(t)) + \sum_{j=2}^{N} X(t) \left( \ln(A)g_{N,j}^{\frac{1}{N}}(A)(1 + L(t)) + \frac{1}{A}(X(t) + D(t)) \right).
\]

Thus, for \(A = c(e + X(t)) \) sufficiently large and \(\varepsilon_1\) small, we obtain

\[
I_1 \leq \frac{D(t)}{16} + c \sum_{j=2}^{N} (e + X(t))g_{N,j}^{2}(e + X(t)) \ln(e + X(t))(1 + L(t))
\]

(23)

by (6). Next, from (17) we work similarly on

\[
I_2 \lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \int_{\mathbb{R}^{N-1}} \|\partial_i u_j\|_{L^{2\gamma N}} \|\nabla b\|_{L^2} \ dx_1 \ldots dx_{N-1}
\]

(24)

\[
\lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left( g_{N,i}^{\frac{1}{\gamma N}}(A) \sqrt{\ln(A)} \|\mathcal{N}_i \partial_i u_j\|_{L^2} + \frac{1}{A} (\|\partial_i u_j\|_{L^2} + \|\partial^2_{N,i} u_j\|_{L^2}) \right) \times \|\nabla b\|_{L^2} \|\nabla \Lambda_{N-1}^{\frac{\gamma N-1}{\gamma N}} b\|_{L^2}
\]

\[
\leq c_2 \|\nabla \Lambda_{N-1}^{\frac{\gamma N-1}{\gamma N}} b\|_{L^2}^2
\]

\[
+ c \|\nabla b\|_{L^2} \left( \sum_{i=1}^{N-1} \sum_{j=2}^{N} g_{N,i}^{\frac{1}{\gamma N}}(A) \ln(A) \|\mathcal{N}_i \partial_i u_j\|_{L^2} + \frac{1}{A} (\|\partial_i u_j\|_{L^2} + \|\partial^2_{N,i} u_j\|_{L^2}) \right)
\]

by Hölder’s inequality, Lemma 2.2 with \(\phi = \partial_i u_j, G_i = g_{N,i}\) of (6), \(\gamma = \frac{1}{2\alpha_{N,j}},\)
Lemma 2.4, Young’s inequality and (18). Identical estimates to (20) lead to

\[
\|\nabla \Lambda_{N-1}^{\frac{\gamma N-1}{\gamma N}} b\|_{L^2}^2 \lesssim X(t) + D(t)
\]

(25)
where the Young's inequalities are justified by (8) and (9). Using (21), (22) and (25) in (24), we obtain

\[ I_2 \leq \frac{D(t)}{16} + c \sum_{j=2}^{N} g_{N,j}^2(e + X(t)) \ln(e + X(t))(e + X(t))(1 + L(t)). \]  

(26)

Next, from (17) we work on

\[ I_3 \lesssim \sum_{j=2}^{N} \int_{\mathbb{R}^{N-1}} \| \partial_N u_j \|_{L^\infty} \| \nabla u \|_{L^2_N}^2 \, dx_1 \ldots dx_{N-1} \]

\[ \lesssim \sum_{j=2}^{N} \int_{\mathbb{R}^{N-1}} \left( g_{N,j}(A) \sqrt{\ln(A)} \| \mathcal{L}_{N,j} u_j \|_{L^2_N} + \frac{1}{\sqrt{A}} \| \partial_N \mathcal{L}_{N,j} u_j \|_{L^2} \right) \| \nabla u \|_{L^2_N} \| \nabla \nabla_{N-1}^{N-1} \ldots \nabla_{N-1}^{N-1} u \|_{L^2} \]

\[ \lesssim \sum_{j=2}^{N} \left( g_{N,j}(A) \sqrt{\ln(A)} \| \mathcal{L}_{N,j} u_j \|_{L^2} + \frac{1}{\sqrt{A}} \| \partial_N \mathcal{L}_{N,j} u_j \|_{L^2} \right) \| \nabla u \|_{L^2} \| \nabla \nabla_{N-1}^{N-1} \ldots \nabla_{N-1}^{N-1} u \|_{L^2} \]

\[ \lesssim \sum_{j=2}^{N} \left( g_{N,j}(A) \sqrt{\ln(A)} \| \mathcal{L}_{N,j} u_j \|_{L^2} + \frac{1}{\sqrt{A}} \| \partial_N \mathcal{L}_{N,j} u_j \|_{L^2} \right) \| \nabla u \|_{L^2} \| \nabla \nabla_{N-1}^{N-1} \ldots \nabla_{N-1}^{N-1} u \|_{L^2} \]

by Hölder's inequality, Lemma 2.3 which is applicable due to (6), Lemma 2.4, Young's inequality and (18). By (20) and the fact that for \( A \) large enough,

\[ \sum_{j=2}^{N} \frac{c \| \nabla u \|_{L^2}^2}{A} \| \partial_N \mathcal{L}_{N,j} u_j \|_{L^2} \leq \sum_{j=2}^{N} \frac{c \| \nabla u \|_{L^2}^2}{A} \| \nabla \mathcal{L}_{N,j} u_j \|_{L^2} \leq \frac{1}{32} D(t) \]

we obtain

\[ I_3 \leq \frac{D(t)}{16} + c \sum_{j=2}^{N} X(t) g_{N,j}^2(e + X(t)) \ln(e + X(t))(e + X(t))(1 + L(t)). \]

(27)
for $\epsilon_3 > 0$ small enough. Similarly, we work on $I_4$ from (17) as follows:

\begin{align}
I_4 & \lesssim \sum_{j=2}^{N-1} \int_{\mathbb{R}^{N-1}} \left| \partial_N u_j \right|_{L^\infty_{x_N}} \left\| \nabla b \right\|_{L_2}^2 d x_1 \ldots d x_{N-1} \\
& \lesssim \sum_{j=2}^{N} \left( g_{N,j}(A) \ln(A) \left\| \mathcal{L}_{N,j} u_j \right\|_{L^2} + \frac{1}{\sqrt{A}} \left\| \partial_N \mathcal{L}_{N,j} u_j \right\|_{L^2} \right) \left\| \nabla b \right\|_{L_2} \left\| \nabla |\nabla_{N-1}^{\mathcal{J}} b \right\|_{L^2} \\
& \leq \epsilon_4 \left\| \nabla |\nabla_{N-1}^{\mathcal{J}} b \right\|_{L^2}^2 + c \left\| \nabla b \right\|_{L_2}^2 \sum_{j=2}^{N} \left( g_{N,j}(A) \ln(A) \left\| \mathcal{L}_{N,j} u_j \right\|_{L^2} + \frac{1}{A} \left\| \partial_N \mathcal{L}_{N,j} u_j \right\|_{L^2} \right) \\
& \leq \frac{D(t)}{16} + c \sum_{j=2}^{N} X(t) g_{N,j}(e + X(t)) \ln(e + X(t))(1 + L(t))
\end{align}

by Hölder’s inequality, Lemma 2.3 which is applicable due to (6), Lemma 2.4, Young’s inequality, (18) and (25). Next,

\begin{align}
I_5 & \lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \int_{\mathbb{R}^{N-1}} \left| \partial_i b_j \right|_{L^\infty_{x_N}} \left\| \nabla u \right\|_{L^2_{x_N}} \left\| \nabla b \right\|_{L^2_{x_N}} d x_1 \ldots d x_{N-1} \\
& \lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left( \sqrt{\ln(A)} t^{\frac{1}{h_{N,j}}} \left( \ln(A) t^{\frac{1}{h_{N,j}}} \left\| \mathcal{L}_{N,j} \partial_i b_j \right\|_{L^2_{x_N}} + \frac{1}{\sqrt{A}} \left\| \partial_i b_j \right\|_{H^1_{x_N}} \right) \right) \\
& \times \left\| \nabla u \right\|_{L^2_{x_N}} \left\| \nabla b \right\|_{L^2_{x_N}} d x_1 \ldots d x_{N-1} \\
& \lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left( \sqrt{\ln(A)} t^{\frac{1}{h_{N,j}}} \left( \ln(A) t^{\frac{1}{h_{N,j}}} \left\| \mathcal{L}_{N,j} \partial_i b_j \right\|_{L^2} + \frac{1}{\sqrt{A}} \left( \left\| \partial_i b_j \right\|_{L^2} + \left\| \partial_i^2 \mathcal{L}_{N,j} b_j \right\|_{L^2} \right) \right) \right) \\
& \times \sqrt{X(t)} \sqrt{X(t) + D(t)} \\
& \leq \epsilon_5 (X(t) + D(t)) + c X(t) \sum_{j=2}^{N} \left( \ln(A) t^{\frac{1}{h_{N,j}}} L(t) + \frac{1}{A} (X(t) + D(t)) \right) \\
& \leq \frac{D(t)}{16} + c \sum_{j=2}^{N} (e + X(t)) h_{N,j}^2 (e + X(t)) \ln(e + X(t))(1 + L(t))
\end{align}

by Hölder’s inequality, Lemma 2.2 with $\phi = \partial_i b_j, G_i = h_{N,j}$ of (8), $\gamma = \frac{1}{27 h_{N,j}},$ Lemma 2.4, (20) and (25). Moreover, we used that similarly to (21), we can show

\[
\sum_{i=1}^{N-1} \sum_{j=2}^{N} \left\| \mathcal{L}_{N} \partial_i b_j \right\|_{L^2} \lesssim 1 + L(t)
\]

by Plancherel theorem and (8) and similarly to (22), using (8) we can obtain

\[
\sum_{i=1}^{N-1} \sum_{j=2}^{N} \left\| \partial_i^2 \mathcal{L}_{N,j} b_j \right\|_{L^2} \lesssim \sum_{i=1}^{N-1} \sum_{j=2}^{N} \left\| \nabla b \right\|_{L^2}^2 + \left\| \nabla |\nabla_{N-1}^{\mathcal{J}} b \right\|_{L^2} \lesssim X(t) + D(t).
\]
Finally,

\[ I_6 \lesssim \sum_{j=2}^{N} \int_{\mathbb{R}^{2N-1}} \| \partial_N b_j \|_{L^2} \| \nabla u \|_{L^2} \| \nabla b \|_{L^2} \ dx_1 \ldots dx_{N-1} \tag{30} \]

\[ \lesssim \sum_{j=2}^{N} \int_{\mathbb{R}^{2N-1}} \left( h_{N,j} \sqrt{\ln(A)} \| M_{N,j} b_j \|_{L^2} + \frac{1}{\sqrt{A}} \| \partial_N M_{N,j} b_j \|_{L^2} \right) \times \| \nabla u \|_{L^2} \| \nabla b \|_{L^2} \ dx_1 \ldots dx_{N-1} \]

\[ \lesssim \sum_{j=2}^{N} \left( h_{N,j} \sqrt{\ln(A)} \| M_{N,j} b_j \|_{L^2} + \frac{1}{\sqrt{A}} \| \partial_N M_{N,j} b_j \|_{L^2} \right) \times \sqrt{X(t)} \sqrt{D(t)} + X(t) \]

\[ \leq C_6 (D(t) + X(t)) \]

\[ + c \sum_{j=2}^{N} X(t) h_{N,j}^2 (e + X(t)) \ln(e + X(t)) L(t) + c X(t) \left( \frac{1}{A} \right) D(t) \]

\[ \leq \frac{D(t)}{16} + e \sum_{j=2}^{N} X(t) h_{N,j}^2 (e + X(t)) \ln(e + X(t))(1 + L(t)) \]

by Hölder’s inequality, Lemma 2.3 which is applicable due to (8), Lemma 2.4, (20), (25), Young’s inequality and (18). By (17), (23) and (26)-(30) we obtain after absorbing \( D(t) \),

\[ \partial_t X(t) + D(t) \tag{31} \]

which implies

\[ \sup_{t \in [0,T]} \int_{e + X(0)}^{e + X(t)} \frac{d\tau}{\sum_{j=2}^{N} \left( g_{N,j}^2 (\tau) + h_{N,j}^2 (\tau) \right) \ln(\tau) \tau} \lesssim \int_0^T (1 + L(\tau)) d\tau \lesssim 1 \]

due to (15). Considering (10), this implies

\[ \sup_{t \in [0,T]} X(t) < \infty \tag{32} \]

Integrating (31) in time, using (32) completes the proof of Proposition 3.2.

**Proof of Theorem 1.1**

We apply \( \Lambda^s \) on (3) and take \( L^2 \)-inner products with \( (\Lambda^s u, \Lambda^s b) \) to obtain

\[ \frac{1}{2} \partial_t Z(t) + E(t) \]

\[ \lesssim \| \Lambda^s u \|_{L^4} \| \nabla u \|_{L^4} \| \Lambda^s u \|_{L^2} + \| \Lambda^s b \|_{L^4} \| \nabla b \|_{L^4} \| \Lambda^s u \|_{L^2} \]

\[ + (\| \nabla u \|_{L^4} \| \Lambda^{s-1} \nabla b \|_{L^4} + \| \Lambda^s u \|_{L^4} \| \nabla b \|_{L^4}) \| \Lambda^s b \|_{L^2} \]

\[ + (\| \nabla b \|_{L^4} \| \Lambda^{s-1} \nabla u \|_{L^4} + \| \Lambda^s b \|_{L^4} \| \nabla u \|_{L^4}) \| \Lambda^s u \|_{L^2} \]

\[ \lesssim \| \Lambda^{s+\frac{1}{5}} u \|_{L^2} \| \Lambda^{\frac{3}{5}} \nabla u \|_{L^2} \| \Lambda^s u \|_{L^2} + \| \Lambda^{s+\frac{2}{5}} b \|_{L^2} \| \Lambda^{\frac{3}{5}} \nabla b \|_{L^2} \Lambda^s u \|_{L^2} \]

\[ + (\| \Lambda^{\frac{2}{5}} \nabla u \|_{L^2} \| \Lambda^{s+\frac{2}{5}} b \|_{L^2} + \| \Lambda^{s+\frac{3}{5}} u \|_{L^2} \| \Lambda^{\frac{2}{5}} \nabla b \|_{L^2}) \| \Lambda^s b \|_{L^2} \]
by Hölder’s inequalities, Lemma 2.5 and by Sobolev embedding of $\dot{H}^{\frac{N}{2}}(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$. We can estimate
\[
\|\Lambda^{s+\frac{N}{2}} u\|_{L^2}^2 \lesssim \sum_{i=1}^{N-1} \sum_{j=1}^N \int |\xi|^{2s} |\xi_j|^{\frac{N}{2}} F(u_j)^2 + \sum_{j=2}^N \int |\xi|^{2s} |\xi_N|^{\frac{N}{2}} F(u_j)^2 + \int |\xi|^{2s} |\xi_N|^{\frac{N}{2}} F(u_1)^2
\]
\[
\lesssim \sum_{i=1}^{N-1} \sum_{j=1}^N \int |\xi|^{2s} (1 + |\xi_i|^{2\alpha_1,1}) F(u_j)^2
\]
\[
+ \sum_{j=2}^N \|\Lambda^{s} \Lambda_N^{\frac{N}{2}} u_j\|_{L^2_{N,j}}^2 + \int |\xi|^{2s} (1 + |\xi_N|^{2\alpha_{N,1}}) F(u_1)^2
\]
\[
\lesssim Z(t) + E(t) + \sum_{j=2}^N \|g_{N,j}(2^k) \frac{2^{k(\frac{N}{2} - \alpha_{N,1})}}{g_{N,j}(2^k)} \| \Delta_k \Lambda^{s} \Lambda_N^{\alpha_{N,1}} u_j\|_{L^2_{N,j}}^2 \|_{L^2_{1}\ldots L^2_{N-1}}^2
\]
\[
\lesssim Z(t) + E(t) + \sum_{j=2}^N \|g_{N,j}(2^k) \frac{2^{k(\frac{N}{2} - \frac{1}{2})}}{g_{N,j}(2^k)} \| \Delta_k \Lambda^{s} \mathcal{L}_N u_j\|_{L^2_{N,j}}^2 \|_{L^2_{1}\ldots L^2_{N-1}}^2
\]
\[
\lesssim Z(t) + E(t)
\]
where $\Delta_k$ is the Littlewood-Paley operator in the $\xi_N$-direction and we used Young’s inequalities justified by (6), (7) and Hölder’s inequality. Similarly we can show
\[
\|\Lambda^{s+\frac{N}{2}} b\|_{L^2}^2 \lesssim Z(t) + E(t),
\]
\[
\|\Lambda^{s} \nabla u\|_{L^2}^2, \quad \|\Lambda^{s} \nabla b\|_{L^2}^2 \lesssim X(t) + D(t).
\]
Therefore, by (18) and Young’s inequalities,
\[
\frac{1}{2} \partial_t Z(t) + E(t) \leq \frac{1}{2} E(t) + cZ(t)[1 + X(t) + D(t)].
\]
Absorbing $E(t)$ and relying on Proposition 3.2 completes the proof of Theorem 1.1.

4. Appendix

Due to similarity to the proof of Theorem 1.1, we only sketch the proof here.

4.1. Proof of Theorem 1.2. We take $L^2$-inner products of (3) with $u$ and $b$ respectively and integrate in time to obtain
\[
\sup_{t \in [0,T]} (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)(t) + \sum_{i,j=1}^N \int_0^T \|\Lambda^{\alpha_{1,j}} u_j\|_{L^2}^2 + \|\Lambda^{\beta_{1,j}} b_j\|_{L^2}^2 dt \lesssim 1. \tag{33}
\]
Now denote by
\[
L(t) \triangleq \sum_{i,j=1}^N \|\Lambda^{\alpha_{1,j}} u_j\|_{L^2}^2 + \|\Lambda^{\beta_{1,j}} b_j\|_{L^2}^2, \tag{34}
\]
\[
D(t) \triangleq \sum_{i,j=1}^N \|\nabla \Lambda^{\alpha_{1,j}} u_j\|_{L^2}^2 + \|\nabla \Lambda^{\beta_{1,j}} b_j\|_{L^2}^2.
\]
Proposition 4.1. Suppose \((u, b)\) solves (3) where \(\{\mathcal{L}_{i,j}\}_{i,j=1}^N, \{\mathcal{M}_{i,j}\}_{i,j=1}^N\) satisfies (11)-(14) in \(\mathbb{R}^N \times [0, T]\), \(N \geq 6, N \in \mathbb{N}\). Then
\[
\sup_{t \in [0, T]} X(t) + \int_0^T D(\tau) d\tau < \infty.
\]

Proof. We have due to Proposition 3.1,
\[
\frac{1}{2} \partial_t X(t) + D(t) \leq \sum_{j=2}^N \left| \nabla u_j \right| |\nabla u|^2 + \left| \nabla u_j \right| |\nabla b|^2 + \left| \nabla b_j \right| |\nabla u| \approx I_1 + I_2 + I_3.
\]

By a similar procedure as in (19) and thereafter, we obtain
\[
I_1 \leq c_1 \left\| \nabla |\nabla_{N-1}|^{\frac{N-1}{2}} u \right\|_{L^2}^2
+ c X(t) \left( \sum_{j=2}^N \ln(A)(|\nabla u_j|_{L^2}^2 + |\nabla_N \nabla u_j|_{L^2}^2) + \frac{1}{A} (|\nabla u_j|_{L^2}^2 + |\nabla_N \nabla u_j|_{L^2}^2) \right)
\]
by Hölder’s inequalities, Lemmas 2.2 and 2.4, Young’s inequalities and (18). Using (11) and (12) we can show
\[
\sum_{j=2}^N \left\| \nabla u_j \right\|_{L^2}^2, \sum_{j=2}^N \left\| \nabla_N \nabla u_j \right\|_{L^2}^2 \lesssim 1 + L(t),
\]
\[
\left\| \nabla |\nabla_{N-1}|^{\frac{N-1}{2}} u \right\|_{L^2}^2, \sum_{j=2}^N \left\| \nabla_N \nabla u_j \right\|_{L^2}^2 \lesssim X(t) + D(t).
\]
These estimates lead to
\[
I_1 \leq \frac{1}{8} D(t) + c(e + X(t)) (\ln(e + X(t))(c + L(t)) + c(1 + L(t)))
\]
if \(A = c(e + X(t))\) for \(c > 0\) large and \(c_1 > 0\) small. Similarly we can show
\[
I_2 \leq \frac{1}{8} D(t) + c(e + X(t)) \ln(e + X(t))(1 + L(t)) + c(1 + L(t)),
\]
\[
I_3 \leq \frac{1}{8} D(t) + c(e + X(t)) \ln(e + X(t))(1 + L(t)).
\]
Taking into account of (36)-(38) in (35), absorbing \(D(t)\) we obtain
\[
\partial_t X(t) + D(t) \lesssim (e + X(t)) \ln(e + X(t))(1 + L(t)).
\]
With Gronwall’s inequality, this completes the proof of Proposition 4.1.

Proof of Theorem 1.2
The \(H^s\)-estimate making use of Proposition 4.1 in this case is completely analogous to that of Theorem 1.1, except easier. We omit further details here.

4.2. Proofs of Lemmas 2.2 and 2.3. We fix \(\gamma \geq 0\) and compute
\[
\left\| \phi \right\|_{L^\infty(\mathbb{R})} \lesssim \sum_{k \geq -1/2^k \leq A} G_1^\gamma (2^k)^{\frac{2^k(\frac{1}{2})}{2^k}} \left\| \Delta_k \phi \right\|_{L^2(\mathbb{R})} + \sum_{k \geq -1/2^k > A} 2^{-k(\frac{1}{2})} 2^k \left\| \Delta_k \phi \right\|_{L^2(\mathbb{R})}
\]
by Littlewood-Paley decomposition and Lemma 2.1. We can continue to bound this last line by

\[
G_i^2(A) \sum_{k \geq -1} \| \Delta_k N_i \phi \|_{L^2(\mathbb{R})} + \left( \sum_{k \geq -1} 2^{-k} \right)^{\frac{1}{2}} \| \phi \|_{H^1(\mathbb{R})}
\]

Due to Plancherel theorem, that fact that \( G_i \) is non-decreasing and Hölder’s inequalities. The second case is a straightforward modification of above. This completes the proof of Lemma 2.2.

Next, we fix \( \gamma \geq \frac{3}{2} \) and compute very similarly

\[
\| \partial_x \phi \|_{L^\infty(\mathbb{R})} \lesssim G_i(A) \sqrt{\ln(A)} \| N_i \phi \|_{L^2(\mathbb{R})} + \frac{1}{\sqrt{A}} \| \partial_x N_i \phi \|_{L^2(\mathbb{R})}
\]

This completes the proof of Lemma 2.3.

4.3. Proof of Lemma 2.4. We estimate

\[
\| \phi \|_{L^4_{x_1} \cdots L^4_{x_{N-1}} L^2_{x_N}} \leq \| \phi \|_{L^2_{x_N} L^4_{x_{N-1}} \cdots L^4_{x_1}} \lesssim \| \phi \|_{L^2_{x_N}} \| \nabla_{x_{N-1}} \phi \|_{L^2_{x_1} \cdots L^2_{x_{N-1}}}
\]

by Minkowski’s inequality for integrals, \((N-1)\)-dimensional Gagliardo-Nirenberg inequality and Hölder’s inequality. This completes the proof of Lemma 2.4.

References