Addendum to Kelley's General Topology

I. Section on Completion of a Uniform Space

Theorem 1. Let $(X, \mathfrak{U})$ be a uniform space. Suppose that we have a dense subset A of X, such that every Cauchy net in A converges to a point of X. Then X is complete – *i.e.*, every Cauchy net in X converges to some point of X.

Proof Let $(x_n)_{n \in D}$ be a Cauchy net in X. Let $E = \{(n, U) \in D \times \mathfrak{U} : i, j \geq n$ in D implies $(x_i, x_j) \in U$. E is a subset of the product directed set $D \times \mathfrak{U}$, where $\mathfrak{U}$ is regarded as being a directed set ordered by reverse inclusion – this is the partial order on $D \times \mathfrak{U}$ such that $(n, U) \geq (n', U')$ **iff** $n \geq n'$ in D and $U' \subset U$. The statement that "the net $(x_n)_{n \in D}$ is a *Cauchy* net" in $(X, \mathfrak{U})$ is equivalent to saying that, "for every $U \in \mathfrak{U}$, there exists $n \in D$ such that (n, U) is an element of E". It follows easily that E is a directed subset of $D \times \mathfrak{U}$, with the induced order. [**Prove this!**]

For each $(n, U) \in E$, we have that $U(x_n)$ is a neighborhood of x_n in X. Since A is a *dense* subset of X, $A \cap U(x_n)$ is non-empty; therefore we can choose an element $y_{(n,U)} \in A \cap U(x_n)$. Choose such an element $y_{(n,U)} \in A \cap U(x_n)$ for every $(n, U) \in E$. (We are here using the Axiom of Choice. It's not difficult to alter our construction so as to avoid using the Axiom of Choice – but we'll avoid complicating our construction by not making this embellishment.)

Then $(y_{(n,U)})_{(n,U) \in E}$ is a net in the subset A of X.

If $(n', U') \geq (n, U)$ and $(n", U") \geq (n, U) \in E$ then $U', U" \subset U$, whence$(y_{(n',U')}, x_{n'}) \in U' \subset U$, and similarly $(y_{(n",U")}, x_{n"}) \in U" \subset U$. Since $(n, U) \in E$ and $n', n" \geq n$ in D, we have that $(x_{n'}, x_{n"}) \in U$. Therefore, $(y_{(n',U')}, y_{(n",U")}) \in U \circ U \circ U^{-1}$. Therefore $(n', U'), (n", U") \geq (n, U)$ in E implies $(y_{(n',U')}, y_{n",U"}) \in U \circ U \circ U^{-1}$. $(n, U) \in E$ being arbitrary, it follows that the net $(y_{(n,U)})_{(n,U) \in E}$ is a Cauchy net in A. By hypothesis, we have that the net $(y_{(n,U)})_{(n,U) \in E}$ converges to some point x in X. We claim that the Cauchy net $(x_n)_{n \in D}$ converges to the same point x in X.

For every $U \in \mathfrak{U}$, since the net $(y_{(n,V)})_{(n,V) \in E}$ converges to x in X, $\exists (n, U) \in E$ such that $(n', U') \geq (n, U)$ in E implies

$$(y_{(n',U')}, x) \in U$$

.

For every $(n', U') \geq (n, U)$ in E, we have that $y_{(n',U')} \in A \cap U'(x'_n)$. Therefore

$$(y_{(n',U')}, x_{n'}) \in U' \subset U$$

.

Therefore

$$(x_{n'}, x) \in U^{-1} \circ U$$

,

all $n' \geq n$ in D. Therefore the net $(x_n)_{n \in D}$ converges to x in X. QED.

Definition 2. Let $(X, \mathfrak{U})$ be a uniform space, and let $U \in \mathfrak{U}$ be an entourage in the uniformity $\mathfrak{U}$. Two Cauchy nets $(x_n)_{n \in D}$ and $(y_m)_{m \in E}$ are *U-close* **iff** the net $(x_n, y_m)_{(n,m) \in D \times E}$ is eventually in U — that is, **iff** $\exists n \in D$ and $m \in E$ such that $i \geq n$ in D and $j \geq m$ in E implies that $(i, j) \in U$. The two nets are *equivalent* **iff** they are U-close, for all $U \in \mathfrak{U}$.

Lemma 3. If $(x_n)_{n \in D}$ and $(y_m)_{m \in E}$ are equivalent Cauchy nets in the uniform space $(X, \mathfrak{U})$, then

1. The set of points to which each of the nets $(x_n)_{n \in D}$ and $(y_m)_{m \in E}$ converges is the same.

2. If $f : (X, \mathfrak{U}) \longrightarrow (Y, \mathfrak{V})$ is a uniformly continuous function, then the Cauchy nets $(f(x_n))_{n \in D}$ and $(f(y_m))_{m \in E}$ in the uniform space $(Y, \mathfrak{V})$ are equivalent.

Proof: Easy.

Corollary 1.1. Let $(X, \mathfrak{U})$ be a uniform space. Suppose that we have a dense subset A of X, such that every Cauchy net in A is equivalent to a Cauchy net in A that converges to a point of X. Then X is complete.

Proof Let $(x_n)_{n \in D}$ be a Cauchy net in A. Then by hypothesis, there exists $(y_m)_{m \in E}$ a Cauchy net in A that is equivalent to $(x_n)_{n \in D}$ such that $(y_m)_{m \in E}$ converges to some point y in X. Hence by the first conclusion of Lemma 3 we have that $(x_n)_{n \in D}$ converges to y in X. $(x_n)_{n \in D}$ being an arbitrary Cauchy net in A, by Theorem 1, we have that the uniform sapce X is complete.

Lemma 4. Let $(x_n)_{n\in D}$ be a Cauchy net in the uniform space $(X, \mathfrak{U})$. For $n \in D$, let $A_n = \{x_i : i \geq n \text{ in } D\}$. Let $\mathfrak{A} = \{A_n : n \in D\}$. Regard $\mathfrak{A}$ as a directed set by reverse inclusion. For every $A \in \mathfrak{A}$, let y_A be an element in A. Then the net $(y_A)_{A\in\mathfrak{A}}$ in X is Cauchy and is equivalent to the Cauchy net $(x_n)_{n\in D}$.

Proof Let $U \in \mathfrak{U}$. Since the net $(x_n)_{n\in D}$ is Cauchy, $\exists n \in D$ such that $i, j \geq n$ in D implies $(x_i, x_j) \in U$. Therefore $A_n \times A_n \subset U$. If $B, C \in \mathfrak{A}$ and $B, C \geq A$ in $\mathfrak{A}$, then $y_B \in B \subset A$ and $y_C \in C \subset A$, whence $(y_B, y_C) \in A \times A \subset U$. Thus, $B, C \geq A$ in $\mathfrak{A}$ implies $(y_B, y_C) \in U$. $U \in \mathfrak{U}$ being arbitrary, we have that the net $(y_A)_{A\in\mathfrak{A}}$ in the uniform space $(X, \mathfrak{U})$ is Cauchy.

Also, if $U \in \mathfrak{U}$ then as above choose $n \in D$ such that $i, j \geq n$ in D implies that $(x_i, x_j) \in U$. Then A_n as above we have that $A_n \in \mathfrak{A}$, and, for any $i \geq n$ in D and any $B \geq A_n$ in $\mathfrak{A}$, we have $y_B \in B \subset A_n$, whence $y_B = x_j$, $\exists j \geq n$ in D. Hence $(x_i, y_B) = (x_i, x_j) \in U$, all $i \geq n$ in D and all $B \geq A_n$ in $\mathfrak{A}$. Therefore the Cauchy nets $(x_n)_{n\in D}$ and $(y_B)_{B\in\mathfrak{A}}$ are equivalent, as asserted.

Proposition 5. Let X be a uniform space. Then there exists a set S of of Cauchy nets in X such that every Cauchy net in X is equivalent to a Cauchy net in S.

Proof. Let S be the collection of all Cauchy nets $(y_B)_{B\in\mathfrak{A}}$ in X such that the directed set $\mathfrak{A}$ is a set of subsets of X ordered by reverse inclusion and such that $y_B \in B$, all $B \in \mathfrak{A}$. Then the collection S is indeed a set, and, by the preceding Lemma, every Cauchy net in the uniform space X is equivalent to one in the set S.

Theorem 6. Let $(A, \mathfrak{U})$ be an arbitrary uniform space. Then there exists a complete uniform space containing A as a dense uniform subspace.

Proof We construct an isomorphism of uniform spaces $\iota : (A, \mathfrak{U}) \longrightarrow (X, \mathfrak{V})$ from $(A, \mathfrak{U})$ onto a dense uniform subspace of a complete uniform space $(X, \mathfrak{V})$. By Proposition 5, there is a set S of Cauchy nets in A such that every Cauchy net in A is equivalent to one in S. $\{0\}$ is a directed set with the trivial ordering. For every $a \in A$, let y_a be the net indexed by the directed set $\{0\}$ that assigns the value a to 0. Then y_a is a Cauchy net in A, and converges to the element $a \in A$, all $a \in A$. Let X be the set consisting of the union of $\{y_a : a \in A\}$ and the set S. Then X is a set of Cauchy nets in A.

For every $U \in \mathfrak{U}$, let U_0 be the set of all pairs $(N, N') \in X \times X$ such that the Cauchy nets N and N' in $(A, \mathfrak{U})$ are U-close. If $V \in \mathfrak{U}$ is symmetric then

$V_0 \subset X \times X$ is symmetric; and if $U, V \in \mathfrak{U}$ are such that $V \circ V \subset U$, then $V_0 \circ V_0 \subset U_0$ [**Proof:** Excercise]. It follows readily that $\{U_0 : U \in \mathfrak{U}\}$ is a base for a uniformity $\mathfrak{V}$ on X. Define $\iota : A \longrightarrow X$ by $\iota(a) = y_a$, all $a \in A$.

If $x = (a_n)_{n \in D}$ is a Cauchy net in $(A, \mathfrak{U})$ in the set S, then we claim that the net $(\iota(a_n))_{n \in D}$ converges to the element $x = (a_n)_{n \in D}$ in the uniform space $(X, \mathfrak{V})$.

We must show that, for every $U \in \mathfrak{U}$, $\exists n \in D$ such that $i \geq n$ in D implies that $(\iota(a_i), x) \in \mathfrak{U}_0$. It is equivalent to say that the Cauchy nets $\iota(a_i)$ and $(a_n)_{n \in D}$ in $(A, \mathfrak{U})$ are U-close.

Since the net $(a_n)_{n \in D}$ in the uniform space $(X, \mathfrak{U})$ is Cauchy, for every $U \in \mathfrak{U}$ there exists $n = n(U) \in D$ such that $i, j \geq n$ in D implies $(a_i, a_j) \in U$. The indexing directed set of the net $\iota(a_i)$ is $\{0\}$. Therefore to show that $\iota(a_i)$ and $(a_n)_{n \in D}$ are U-close we must show that there exists $n \in D$ such that $j \geq n$ in D implies that $(a_i, a_j) \in U$. Taking $n = n(U)$ as above, we therefore have that, for $i \geq n$ in D the nets $\iota(a_i)$ and $(a_n)_{n \in D}$ are indeed U-close, since for $j \geq n$ $(a_i, a_j) \in U$.

Therefore we have shown that, for every Cauchy net $(a_n)_{n \in D}$ in S, the net $\iota(a_n)_{n \in D}$ converges in X. By Corollary 1.1, to complete the proof of the Theorem, it suffices to prove that $\iota(A)$ is dense in X. But if $x \in X$, then either $x \in \iota(A)$ or $x \in S$. In the latter case, we have that $x = (a_n)_{n \in D}$, a Cauchy net in A that is in the set S. But then, in the preceding paragraph, we constructed a net in $\iota(A)$ that converges to x in X. Therefore $\iota(A)$ is dense in X.

Theorem 7. (Version of Kelley, Theorem 26, pg. 195.) Let A be a dense subset of the uniform sapce $(X, \mathfrak{U})$; regard (as always) A as being a uniform space with the induced uniformity $\mathfrak{U}_A$ from $(X, \mathfrak{U})$. Suppose that

$$f : (A, \mathfrak{U}_A) \longrightarrow (Y, \mathfrak{V})$$

is a uniformly continuous function from A into a complete Hausdorff uniform spave $(Y, \mathfrak{V})$. Then $\exists!$ continuous extension

$$\overline{f} : (X, \mathfrak{U}) \longrightarrow (Y, \mathfrak{V})$$

of f. And $\overline{f}$ is uniformly continuous.

Proof If $x \in X$, then since $x \in \overline{A}$, there is a net $(x_i)_{i \in D}$ in A that converges to x in X. Hence the net $(x_i)_{i \in D}$ in A is Cauchy. Since f is uniformly continuous, the net $f(x_i)_{i\in D}$ in Y is Cauchy. Since Y is complete and Hausdorff, the Cauchy net $f(x_i)_{i\in D}$ in Y converges to a unique element of Y; call it $\overline{f}(x)$.

This definition of $\overline{f}(x)$ is independent of the net $(x_i)_{i\in D}$ in A chosen converging to x in X: Suppose $(x'_j)_{j\in E}$ is another net in A converging to the same element x in X. Then since the nets $(x_i)_{i\in D}$ and $(x'_j)_{j\in E}$ converge to the same element $x \in X$, these Cauchy nets in A are equivalent. Since $f : A \longrightarrow Y$ is uniformly continuous, the nets $f(x_i)_{i\in D}$ and $f(x'_j)_{j\in E}$ in X are equivalent Cauchy nets, and therefore converge to the same element $y \in Y$. Therefore we have a well-defined function $\overline{f} : X \longrightarrow Y$.

Next we show that if $W \in \mathfrak{V}$ then there is a $U \in \mathfrak{U}$ such that $\overline{f} \circ U \subset W \circ \overline{f}$ – this is equivalent to saying that $\overline{f}$ is uniformly continuous, completing the proof.

Choose $V \in \mathfrak{V}$ closed and symmetric such that $V \circ V \subset W$, and choose $U \in \mathfrak{U}$ open and symmetric such that $f(U(a)) \subset V(f(a))$ for all $a \in A$ (can do, since $f : A \longrightarrow Y$ is uniformly continuous). If $(x, u) \in U$ and $\overline{f}(x) = y$ and $\overline{f}(u) = v$ then $U(x) \cap U(u)$ is open (and non-empty since it contains both x and u). Since A is dense in X, $\exists z \in A \cap U(x) \cap U(u)$. Then $x, u \in U(z)$. Therefore $y = \overline{f}(x) \in \overline{f(U(z))}$ (the latter meaning the closure of $f(U(z))$). [**Proof:** $x \in U(z)$. Since $y = \overline{f}(x)$, there is a net $(x_i)_{i\in D}$ in A converging to x in X such that the net $(f(x_i))_{i\in D}$ converges to y in Y. $U(z)$ is an open set in X containing x; therefore the net $(x_i)_{i\in D}$ is eventually in $U(z)$. Therefore the net $(f(x_i))_{i\in D}$ is eventually in $f(U(z))$. Since $f(x_i)_{i\in D}$ converges to y in Y, we have that $y \in \overline{f(U(z))}$.] Similarly, $v = \overline{f}(u) \in \overline{f(U(z))}$.

So $y, v \in \overline{f(U(z))} \subset \overline{V(f(z))}$ [This last inclusion since $f(U(z)) \subset V(f(z))$, since $z \in A$, and $f(U(a)) \subset V(f(a))$, for all $a \in A$.]

Therefore $y, v \in \overline{V(f(z))} = V(f(z))$, since V is a closed entourage in Y. Therefore $(y, v) \in V \circ V \subset W$. Therefore indeed if $(x, u) \in U$ then $(\overline{f}(x), \overline{f}(u)) \in W$, as asserted, whence $\overline{f}$ is uniformly continuous.

II. Interpretaion of part of Kelley's Chapter on Function Spaces

Let X be a set and let $(Y, \mathfrak{U})$ be a uniform space. Then Y^X has the *product uniformity*, since $Y^X = \prod_{x\in X} Y$. This is also called the *uniformity of pointwise convergence*. The function $e_x : Y^X \longrightarrow Y$, *evaluation at* x, is the projection $\pi_x : \prod_{x\in X} Y \longrightarrow Y$ onto the x'th coordinate – that is, $e_x : Y^X \longrightarrow X$ is the function, $e_x(f) = f(x)$, all $x \in X$.

The uniformity of pointwise convergence can be characterized as being the coarsest uniformity on the set Y^X rendering the evaluation map $e_x : Y^X \longrightarrow X$ uniformly continuous for all $x \in X$. (Note that this observation is a special case of the more general observation that we made when we discussed the product uniformity on $\prod_{x \in X} Y_x$, where Y_x is a uniform space, for all $x \in X$.)

A subbase for the uniformity of pointwise convergence on Y^X are the entourages $V_{x,U}$ for all $x \in X$ and all $U \in \mathfrak{U}$, where

$$V_{x,U} = \{(f, g) \in Y^X \times Y^X : (f(x), g(x)) \in U\}.$$

A net of functions $(f_n)_{n \in D}$ in Y^X converges to a function f in Y^X for the topology of pointwise convergence **iff** the net in Y $(f_n(x))_{n \in D}$ converges to $f(x) \in Y$ for all $x \in X$ – [which it is why this topology on Y^X is called *the topology of pointwise convergence.*]

And a net $(f_n)_{n \in D}$ of functions from X into Y is Cauchy for the uniformity of pointwise convergence **iff** the net in the uniform space Y $(f_n(x))_{n \in D}$ is Cauchy, for every x in X.

Again, if X is a set and $(Y, \mathfrak{U})$ a uniform space, than a *finer* uniformity on the set Y^X is the *uniformity of uniform convergence.* For each $U \in \mathfrak{U}$, let $U_0 = \{(f, g) \in Y^X \times Y^X : (f(x), g(x)) \in U$ for *all* $x \in X\}$. Thus, $U_0 = \cap_{x \in X} V_{x,U}$, and in particular $U_0 \subset V_{x,U}$, all $U \in \mathfrak{U}$, all $x \in X$.

Then [**prove!**] for $U, V \in \mathfrak{U}$

1. $(U_0 \circ V_0) \subset (U \circ V)_0$,
2. $U \subset V \Longrightarrow U_0 \subset V_0$,
3. $(U \cap V)_0 = U_0 \cap V_0$,
4. $(U_0)^{-1} = (U^{-1})_0$, and
5. $\Delta_{Y^X} \subset U$.

It follows that $\mathfrak{U}_0 = \{U_0 : U \in \mathfrak{U}\}$ is the base for a uniformity on Y^X, called the *uniformity of uniform convergence.*

Recall: If Z is any set and $\mathfrak{B}$ is any collection of subsets of $Z \times Z$ then $\mathfrak{B}$ is the base for a uniquely determined uniformity on Z **iff**

1. $U \in \mathfrak{B} \Longrightarrow \exists V \in \mathfrak{B}$ such that $V \circ V \subset U$,

2. $U, V \in \mathfrak{B} \Longrightarrow \exists W \in \mathfrak{B}$ such that $W \subset U \cap V$,

3. $U \in \mathfrak{B} \Longrightarrow \exists V \in \mathfrak{B}$ such that $V^{-1} \subset U$

4. $U \in \mathfrak{B} \Longrightarrow \Delta_Z \subset U$.

To verify the first of these conditions for $\mathfrak{U}_0$: If $U \in \mathfrak{U}$, choose $V \in \mathfrak{U}$ such that $V \circ V \subset U$. Then $V_0 \circ V_0 \subset (V \circ V)_0 \subset U_0$.

Since $U_0 = \cap_{x \in X} V_{x,U}$ for all $U \in \mathfrak{U}$, we have that the uniformity of uniform convergence is finer than the uniformity of pointwise convergence.

The topology of the uniformity of uniform convergence is called the *topology of uniform convergence*; it depends on the uniformity $\mathfrak{U}$ of Y, not just the topology of Y. [On the other hand, the topology of pointwise convergence on Y^X depends only on the topology of Y; and of course makes sense when Y is just a topological space, not a uniform space.]

Theorem. If X is a topological space and $(Y, \mathfrak{U})$ is a uniform space, then the set $\mathfrak{C}$ of all continuous functions $f : X \longrightarrow Y$ from X into Y is closed in Y^X for the topology of uniform convergence.

Note: It is equivalent to say that, if D is a directed set and if $f_n : X \longrightarrow Y$ is a continuous function from X into Y for all $n \in D$, and if the net $(f_n)_{n \in D}$ converges to a function $f \in Y^X$ *for the topology of uniform convergence* [Equivalent terminology: "and if the net $(f_n)_{n \in D}$ of continuous functions from X into Y *converges uniformly* to a function $f : X \longrightarrow Y$"], then the function $f : X \longrightarrow Y$ is continuous.

Proof. For any $x_0 \in X$, we show that f is continuous at x_0. To show this, we must show that, for every $U \in \mathfrak{U}$, there exists a neighborhood V of x_0 in X such that $f(V) \subset U(f(x_0))$.

Since $(f_n)_{n \in D}$ converges uniformly to f in Y^X, we know that $\exists N \in D$ such that

(i) $(f_i(x), f(x)) \in U$, for all $i \geq N$ in D and all $x \in X$.

In particular,

(i') $(f_N(x), f(x)) \in U$, for all $x \in X$.

Since $f_N : X \longrightarrow Y$ is continuous, it is continuous at x_0. Therefore there exists a neighborhood V of x_0 in X such that $f_N(V) \subset U(f_N(x_0))$. That is, we have that

(ii) $(f_N(v), f_N(x_0)) \in U$, for all $v \in V$.

Equation (i') hold for all $x \in X$, and in particular for all $v \in V$:

(iii) $(f_N(v), f(v)) \in U$, for all $v \in V$.

Since $x_0 \in V$, (iii) implies

(iv) $(f_N(x_0), f(x_0)) \in U$.

If $U \in \mathfrak{U}$ is symmetric, then (ii), (iii) and (iv) imply that

$(f(v), f(x_0)) \in U \circ U \circ U$ for all $v \in V$.

[(iii) says that "$f(v)$ and $f_N(v)$ are U-close"; (ii) says that "$f_N(v)$ and $f_N(x_0)$ are U-close"; and (iv) says that "$f_N(x_0)$ and $f(x_0)$ are U-close"].

Thus, for every symmetric entourage $U \in \mathfrak{U}$, we've found a neighborhood V of x_0 in X such that

$f(V) \subset (U \circ U \circ U)(f(x_0))$.

Therefore $f : X \longrightarrow Y$ is continuous at x_0. This being the case for every $x_0 \in X$, we have that $f : X \longrightarrow Y$ is continuous.

Example. Let $f_n : [1, \infty) \longrightarrow \mathbb{R}$ be the continuous function, $f_n(x) = \frac{1}{x_n}$, $x \geq 1$. Then the sequence of functions $(f_n)_{n \geq 1}$ converges *pointwise* to the function $f : X \longrightarrow Y$, where

$$f(x) = \begin{cases} 1, & x = 1 \\ 0, & x > 1, \end{cases}$$

a function that is not continuous.

Thus, to ensure continuity for the limit of a net of continuous functions from a topological space to a uniform space, we need to know that the net *converges uniformly*, not merely that it converges pointwise.

Note: If X is a set and Y is a pseudometric space, and if $(f_n)_{n \in D}$ is a net of functions from the set X into the pseudometric space (Y, d), and if $f : X \longrightarrow Y$ is a function, then the net $(f_n)_{n \in D}$ converges uniformly to f **iff**

for every $\epsilon \geq 0, \exists N = N(\epsilon) \in D$ such that

$$d(f_n(x), f(x)) \leq \epsilon \text{ for all } n \geq N \text{ and for all } x \in X.$$

The net $(f_n)_{n \in D}$ converges pointwise to f **iff** for every $x \in X$ and every $\epsilon \geq 0, \exists N = N(x, \epsilon) \in D$ such that

$$d(f_n(x), f(x)) \leq \epsilon \text{ forall } n \geq N$$

Note also: If X is a set and $(Y, \mathfrak{U})$ is a uniform space, then a net of functions $f_n : X \longrightarrow Y$, $n \in D$ is *Cauchy for the uniformity of pointwise convergence* **iff** the net $(f_n(x))_{n \in D}$ in the uniform space $(Y, \mathfrak{U})$ is a Cauchy net, for all $x \in X$ **iff** for every $x \in X$ and every $U \in \mathfrak{U}, \exists N = N(x, U) \in D$ such that $i, j \geq N$ in D implies $(f_i(x), f_j(x)) \in U$.

The net of functions $f_n : X \longrightarrow Y$, $n \in D$ is *Cauchy for the uniformity of uniform convergence* **iff** for every $U \in \mathfrak{U}, \exists N = N(U) \in D$ such that $i, j \geq N$ in D implies $(f_i(x), f_j(x)) \in U$ for all $x \in X$.