MATH 557: TOPICS IN DIFFERENTIAL GEOMETRY

Course Description

The course will be divided roughly into two parts:

- In the first part, we will discuss the basic propeties of Sobelev spaces that are needed for standard treatments of existence and basis regularity of linear elliptic PDE. This part of the class will follow Lieb and Loss's Analysis for the basic properties of L^P spaces and Evan's Partial Differential Equations Chapters 5 and 6 for the $W^{k,p}$ spaces.
- In the second part of the class, we will the basic theory of minimal surfaces and survey some of the classic results. Here, we will follow closely Colding-Minicozzi's book A Course in Minimal Surfaces. Beyond the basic thoery, the topics we will discuss include: The classical Plateau Problem via the Dirichlet problem for harmonic functions, existence and regularity theory for harmonic maps in two dimensions.

Basic plan

We have 28 class days. The first day of classes is 1.18 and the last is 5.3. Spring break is 3.5-3.11. This leaves us with 14 Tuesdays and 14 Thursdays. I have the following budgeting of time in mind for the various topics:

- Sobelev spaces and basic PDE theory. Overall, roughly 14 days should be enough, with the following break down:
  - L^p spaces and measure theory (convergence theorems, inequalities, separability, density of smooth functions, Banach-Alaoglu theorem, convolution products): Days 1-3
  - $W^{k,p}$ spaces (Weak derivatives, Approximations by smooth functions, Extensions and Traces, Sobelev Inequalities, Rellich-Compactness): Days 4-10.
  - Basic existence and regularity theory for linear elliptic PDE: Days 10-16
- Topics in minimal surface theory: The remaing approximately 14 class days. As we approach this time in the class I will make a more detailed outline of how we plan budget our time on minimal sufaces and harmonic maps.

Time line

Day 1 (R 1.19). We had a brief group discussion to gauge the interests of the audience. Determined that most of the class would not mind a review of basic Sobelev space theory before discussing existence and regularity questions for second order linear elliptic PD

Day 2 (T 1.24). Began discussing basic properties of the Sobelev spaces $W^{k,p}$, following Evan's PDE. We covered most of Secton 5.2, including the definition of weak derivatives and their properties. We gave examples of regularity issues in these spaces. In particular, we gave an example of a function in$W^{1,p}$ which isn't even locally bounded! Strange stuff indeed

Day 3 (R 1.26). We went further back in time to study L^p spaces right from the bottom. For this we are following Lieb and Loss' Analysis. We did defined measureable functions–for this we needed to define the notion of measure space–summable functions, and reviewed the basic convergence theorems: Monotone and Dominate. We also proved Jensen's Inequailty–which relates averages and convex functions in meausre spaces–and Holder's Inequality from Jensen's. We finished by proving that L^p spaces are complete. Our proof relied on both the Monotone and Dominated Convergence Theorems.
Projecting into the future: Before continuing, we will need know the following things about L^p spaces, which will probably take an additonal one to two days of class time:

- L^p spaces are separable: That is, there is a countable basis. This is a critically important property.
- Compactly supported smooth function are dense in L^p. In practice, it is extremely convenient to prove things about Sobelev spaces using smooth compactly supported functions and density arguments

The Banach-Alaoglu Theorem: Bounded sequences have weak limits.

Day 4(T 1.31). Continuation of basic L^p theory: We ended up discussing the density of smooth functions on L^p spaces. In particular, we discussed the fact that the problem approximating functions in L^p by smooth functions and be reduced to approximating characteristic functions of measurable set, which can then be reduced to approximating characteristing functions of rectangles–this reduction follows the treatment in Lieb and Loss's Analysis. We didn't do this in all detail, and below we will have some exercises on this reduction process. To show that characteristic functions can be approximated by smooth functions, we proved Young's Inequality and used it to show convolutions of L^p functions with L^1 functions are still in L^p. It remains to show

(1) Convolutions with smooth functions are smooth and $D(f * j) = f * (Dj)$ (in the case that j is smooth)

(2) Convergence in L^p of $f_\epsilon = f * j_\epsilon$ to f in L^p when $f = \mathcal{X}_\mathcal{A}$ is the characteristic function of a rectangle

We will show (2) above in class in the special case that both f and j are compactly supported on $\mathbb{R}^n$. The reduction to this case and (1) above will be left as exercise.

Exercises for Day 4. From Lieb and Loss: Ch1: 9, Ch2: 2, 7, 17. Additionally:

Problem 1. (approximating summable functions by simple functions) A simple function is a measurable function that takes a finite number of values, i.e. a linear combination of characteristic functions of measurable sets. Show that given a summable function f and ϵ positive, there is a simple function g such that $\int_\Omega |f-g| d\mu \leq \epsilon$

Problem 2. (approximating simple functions by really simple functions) (WARNING: For the ensuing problems, assume this result. The proof requires sort of a large digression back into meassure theory, which we will postpone untile later). Given a measureable set A and $\epsilon > 0$ show that there is a finite collection of boxes (products of interevals) whose union R satisfies $\mu(A \Delta R) \leq \epsilon$. Here, Δ denotes the symmetric difference operator ons sets.

Problem 3. (convolutions of rectangles). Let R be a rectangle and let χ_R be its characteristic function. Set $f_\epsilon := \chi_R * j_\epsilon$, where j_ϵ is as in class. Show that $f_\epsilon \to \chi_A$ as $\epsilon \to 0$ is L^p

Problem 4. Combine Problems 1-3 to to show that: Given $f \in L^p(\mathbb{R}^n)$ and $\epsilon > 0$ there is $g \in C^{\infty_c}(\mathbb{R}^n)$ such that $\|f-g\|_p \leq \epsilon$.

Day 5 (R 2.02). We finished our discussion of density of smooth functions in L^p spaces. Additionally, we showed that L^p spaces are separable, with the same countable basis independent of p. The basis was explicitly constructed as essentially step functions with rational values on rectangles. Finally, we breifly discussed the definition of weak convergence and mentioned Riesz Represention Theorem: The dual space to L^p is $L^{p'}$, $1 < p < \infty$.
Regarding density of smooth functions. Our treatment followed Lieb and Loss: Previously we had shown using Young's Inequality that if f belongs to L^p and j belongs to L^1 then the convolution product $f * j$ belongs to L^p and we have the estimate $\|f * j\|_p \leq \|j\|_1 \|f\|_p$. Our remaing step was to show that the family of renormalized convolutions $f_\epsilon := f * j_\epsilon$ converge back to f in L^p. We argued that it suffices to to assume that f is bounded and with compact support–essentially applications of the Dominated Convergence Theorem. Compactly supported functions are in L^1 and we have $\|f\|_1 \leq K\|f\|_p$ with a constant that depends on the support of f. We showed previously, that L^1 functions can be norm approximated by really simple functions. Pick a simple function F such that $\|f - F\| \leq \delta$. We can assume that both$F$ and$f$ had the same δ independent support. The triangle inequality, Holder's and Young's Inequality gives

$$\|f_\epsilon - f\|_p \leq \|f - F\|_p + \|(f - F) * j_\epsilon\|_p + \|F_\epsilon - F\|_p \leq K\delta + K\|j\|_1\delta + K\|F_\epsilon - F\|_1$$

Exercises for Day 5.

Problem 5. (Linear functionals separate) Let $f$be in $L^p(\Omega)$ and assume that $L(f) = 0$ for every bounded linear functional L on L^p. Show that $f = 0$. (Hint: Set $g = f^{p-1}$. Show that $\int fg = \|f\|_p^p$. Show that g is in $L^{p'}$)

Problem 6.

Day 6 (T 2.07). We will finish with our treatment of L^p the a discussion of Riesz Representation Theorem and The Banach-Alaoglu Theorem: L^p bounded sequences have weak limits. We will not provide all the details and some will be left as excercises. When this is finished, we will discuss briefly approximations by smooth functions in $W^{k,p}$ spaces. On domains that are compact, there are the separate types of approximations:

(1) Local: That is, on compact subsets of of the domain.
(2) Global interior: Approximation in $W^{k,p}$ by smooth functions on the interior of the domain
(3) Up to the boundary: Approximation by smooth functions which are uniformly bounded in C^k on every compact subset of the domain–e.g. the modulus of continity doesn't blow up as you approach the boundary.

The basic idea is to convolve with approximate identities (smooth compactly supported functions which integrate to 1). This essentially immediately gives (1). However, for (2) and (3) there are some technical issues having to do with the boundary. It is not clear in how much depth we will discuss this. Some points will be left as an excercise.

Remark. We have yet to finish point (3) above. That is, we haven't yet showed that, in the case that the boundary of the domain is C^1, functions that are smooth up to the boundary are dense in $W^{k,p}$. This of course will spill into day 7. Additionally, we will for the purposes of this class, assume the Riesz Representation Theorem without proving it. You may see some exercises on the topic later though.

Exercises for Day 6.

Problem 7. In the proof of the Banach-Alaoglu Thereom, we proceeded in two parts: Considering a bounded sequence in L^p, we obtained a subsequence g_k such that the integrals $\int g_k\phi$ converge to a limit as k goes to infinity for each ϕ in our dense subset. This was an application of Cantor's Idea. In this way, we defined a function L on this dense subset as follows: $L(\phi) := \lim_{k\to\infty} \int g_k\phi$. Since the subset is dense, we can try to extend the definition of L to all of $L^{p'}$ by: $L(h) = \lim_{i\to\infty} L(\phi_i)$ where $\phi_i \to h$ is an approximating sequence: Show first that this is well defined by taking a different approximating sequence and showing that the answers agree. Show that the result is a continuous function on L^p. Show that it is linear

Problem 8. Show that a bounded linear operator is the same thing as a continuous linear operator on L^p spaces.

Problem 9. From the previous problem, it follows that the operator in the previous exercise is bounded, and hence belongs to the dual of $L^{p'}$. Conclude Banach-Aalaoglu from Riesz.

Day 7 (R 2.09). We will finish our discussion of approximation of smooth functions in Sobelev spaces. We will then discuss extensions and traces and characterize the space $W_0^{k,p}$-define as the closure as smooth compactly supported functions in the sobelev norm as those with zero trace. Time permitting we will start the discussion of Sobelev Inequalities.

Day 8 (T 2.14). Murat gave a very nice, detailed explanation of extensions in $W^{1,p}$. It was lovely to see all the details handled so carefully. A few questions arose along the way

(1) Why does the extension map only give equality almost everywhere, and not everywhere?
(2) Partitions of unity: Why is there a smooth partition of unity for every locally finite open cover?

On Thursday, we will continue with our characterization of the space $W_0^{1,p}$–My lecture on this was admittedly not the best so it is worth going back over it–and once this is finished we will start the new topic of Sobelev Inequalities.