Math 531 Tom Tucker
NOTES FROM CLASS 10/13

We were in the middle of proving the following...

Lemma 19.1. Let $A$ and $B'$ be as last time. Let $\mathcal{P}$ be a maximal prime of $A$, let $k = A/\mathcal{P}$, let $S = A \setminus \mathcal{P}$, and let $\phi : S^{-1}B' \to S^{-1}B'/S^{-1}B'\mathcal{P}$ be the usual quotient map. Let us denote $S^{-1}B'/S^{-1}B'\mathcal{P}$ as $C$. Then for any $y \in S^{-1}B'$, we have $\phi(T_{L/K}(y)) = T_{C/k}(\phi(y))$.

Proof. Let $\bar{w}_1, \ldots, \bar{w}_n$ be a basis for $C$ over $k$ and pick $w_i \in B'$ such that $\phi(w_i) = \bar{w}_i$. Since the $\bar{w}_i$ are linearly independent, the $w_i$ must be as well. To see this, suppose that $\sum_{i=1}^n a_i w_i = 0$ for $a_i \in S^{-1}B'$ (remember that everything in $L$ is $x/a$ for $x \in B'$ and $a \in A$). By dividing through by a power of a generator $\pi$ for $A\mathcal{P}$, we can assume that not all of the $a_i$ are in $S^{-1}B'\mathcal{P}$. This means then that $\sum_{i=1}^n \phi(a_i) \bar{w}_i = 0$, with some $\phi(a_i) \neq 0$, which is impossible. Now, we are essentially done, since we can define the trace of any $y \in B'$ with respect to this basis. We have

$$yw_i = \sum_{j=1}^n m_{ij}w_j$$

with $m_{ij} \in A$, and

$$\phi(y)\bar{w}_i = \sum_{j=1}^n \phi(m_{ij})\bar{w}_j.$$ 

Hence,

$$\phi(T_{L/K}(y)) = \sum_{i=1}^n \phi(m_{ii}) = T_{C/k}(\phi(y)).$$

□

We need one quick lemma from linear algebra.

Lemma 19.2. Let $V$ be a vector space. Let $\phi : V \to V$ be a linear map. Suppose that $\phi^k = 0$ for some $k \geq 1$. Then the trace of $\phi$ is zero.

Proof. This is on your HW. □

When $B$ is the integral closure of $A$ in $L$, and $\mathcal{P}$ is maximal in $A$, we can write

$$\mathcal{P}B = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$ 

If $e_i > 1$ for some $i$, then we say that $\mathcal{P}$ ramifies in $B$. When $B = A[\alpha]$, we know that $\mathcal{P}$ ramifies in $B$ if and only if $\Delta(B/A) \subseteq \mathcal{P}$. That is true more generally.
Theorem 19.3. Let $B$ be the integral closure of $A$ in $L$ and let $\mathcal{P}$ be maximal in $A$. Then $\mathcal{P}$ ramifies in $B$ if and only if $\Delta(B/A) \subseteq \mathcal{P}$.

Proof. It will suffice to prove this locally, that is to say, it will suffice to replace $A$ with $A_\mathcal{P}$ and $B$ with $S^{-1}B$ where $S = A \setminus \mathcal{P}$. As in the previous Lemma, we write $k = A/\mathcal{P}$ and $C = S^{-1}B/\mathcal{P}S^{-1}B$ and let

$$\phi : S^{-1}B \longrightarrow S^{-1}B/\mathcal{P}S^{-1}B$$

Also, as in that Lemma let $\tilde{w}_1, \ldots, \tilde{w}_n$ be basis for $C$ over $k$ and pick $w_i \in S^{-1}B$ such that $\phi(w_i) = \tilde{w}_i$. It is clear then that

$$A_\mathcal{P}w_1 + \ldots + A_\mathcal{P}w_n + \mathcal{P}S^{-1}B = S^{-1}B,$$

so by Nakayama’s Lemma, the $w_i$ generate $S^{-1}B$ as an $A_\mathcal{P}$ module. From the Lemma above we have $T_{L/K}(w_i w_j) = T_{C/k}(\tilde{w}_i \tilde{w}_j)$, so the matrix $M = [T_{C/k}(\tilde{w}_i \tilde{w}_j)]$ represents the form $(x, y) = T_{C/k}(xy)$ on $C/k$. Let us now decompose $C/k$ as ring, we have

$$C \cong S^{-1}B/\mathcal{P}S^{-1}B \cong \bigoplus_{i=1}^m S^{-1}B/S^{-1}BQ_i^e$$

where

$$\mathcal{P}B = Q_1^{e_1} \cdots Q_m^{e_m}.$$ 

If $e_i > 1$, then any element $z \in C$ such that $z = 0$ in every coordinate but $i$ and has $i$-th coordinate in $Q_i$, has the property that $z^{e_i} = 0$. This means that the pairing

$$(x, y) = T_{C/k}(xy)$$

on $C$ is degenerate from your homework. If $e_i = 1$ for every $i$, then

$$C \cong S^{-1}B/S^{-1}BQ_1 \oplus \cdots \oplus S^{-1}B/S^{-1}BQ_m$$

and $S^{-1}B/S^{-1}BQ_i$ is separable over $k$ for each $i$. The trace form $(x, y) = T_{C/k}(xy)$ decomposes into a sum of forms

$$(a, b) = T_{(S^{-1}B/S^{-1}BQ_i)/k}(ab),$$

each of which is nondegenerate, so $(x, y)$ is nondegenerate, so

$$\det[T_{L/K}(w_i w_j)] \notin \mathcal{P},$$

and we are done. $\square$

Here is a simple and easy to prove fact comparing the discriminants of different subrings $B$ and $B'$ of $L$.
Proposition 19.4. Let $B' \subset B$ where $B$ and $B'$ are as usual (we will usually take $B$ to be the integral closure of $A$ in $L$). Suppose that $B$ has a basis $v_1, \ldots, v_n$ as an $A$-module and that $B'$ has a basis $w_1, \ldots, w_n$ as an $A$-module. Writing

$$w_i = \sum_{\ell=1}^n n_{i\ell} a_{\ell},$$

and letting $N$ be the matrix $[n_{i\ell}]$, we have

$$\det[T_{L/K}(w_i w_j)] = (\det N)^2 \det[T_{L/K}(v_i v_j)].$$

Proof. Now,

$$T_{L/K}(w_i w_j) = \sum_{\ell=1}^n \sum_{k=1}^n n_{i\ell} n_{jk} T_{L/K}(v_i v_j).$$

A bit of linear algebra shows that this is exactly the same as the $ij$-th coordinate of the matrix $N^t M N$ where $M = [T_{L/K}(v_i v_j)]$. Equation 1 follows. I gave an easier explanation on the board. □