Next, we will show that $\ell(\mathcal{O}_x^*)$ is a sublattice in $\mathbb{R}^{r+s}$. We define a sublattice as a subgroup of $\mathbb{R}^m$ that has $\mathbb{Z}$-rank equal to the $\mathbb{R}$-dimension of the vector space it generates.

We were in the middle of proving the following.

**Proposition 33.1.** Let $\mathcal{L}$ be a finitely generated subgroup of $\mathbb{R}^m$. Then $\mathcal{L}$ is a sublattice if and only if every bounded region in $\mathbb{R}^m$ contains at most finitely many elements of $\mathcal{L}$.

**Proof.** Note, we already proved the “only if” part last week during our proof of the finiteness of the class group.

We will prove the “if” part by induction on $m$. If $m = 1$ and $\mathcal{L} \neq 0$ (0 is trivially a sublattice), then $\mathbb{R}^m = \mathbb{R}$, and we choose $u$ to be the smallest positive number in $\mathcal{L}$. Then, for any $v \in \mathcal{L}$, we can write $v = tu + z$ where $t$ is an integer and $0 \leq z < u$. But, since $z = v - tu$, we must have $z \in \mathcal{L}$, which means that $z = 0$ by the minimality of $u$. Thus, $u$ must generate $\mathcal{L}$ as a $\mathbb{Z}$-module, so the rank of $\mathcal{L}$ as a group is equal to 1.

Now, we do the inductive step. Note that we may assume $\mathcal{L}$ generates $\mathbb{R}^m$ as a vector space, since otherwise it is contained in a vector space of dimension $\mathbb{R}^{m-1}$ and we are done by the inductive hypothesis. Thus, we can choose $\mathbb{R}$-linearly independent elements $v_1, \ldots, v_m$ of $\mathcal{L}$. By the inductive hypothesis, if $V_0$ is the $\mathbb{R}$-vector space generated by $v_1, \ldots, v_{m-1}$, then $\mathcal{L}_0 := V_0 \cap \mathcal{L}$ is a sublattice, and is a full lattice in $V_0$. Let $w_1, \ldots, w_{m-1}$ be a basis for $\mathcal{L}_0$ (as a $\mathbb{Z}$-module). Then, $w_1, \ldots, w_{m-1}, v_m$ is a basis for $\mathbb{R}^m$, so any element of $\lambda \in \mathcal{L}$ can be written as

$$\lambda = \sum_{i=1}^{m-1} r_i w_i + r_m v_m$$

for real numbers $r_i$. Note that if $r_m = 0$, then $\lambda \in \mathcal{L}_0$, and we can choose all of the $r_i$ to be integers. Note also that by subtracting off an appropriate element of $\mathcal{L}_0$, we obtain such a $\lambda$ with all $0 \leq r_i < 1$ for $i \leq (m - 1)$. There are only finitely many such $\lambda$ with $r_m$ also smaller than a certain bound (since any bounded region in $\mathbb{R}^m$ intersects $\mathcal{L}$ in finitely many points). Thus, there is a nonzero element $\lambda'$ with $0 \leq r_i < 1$, for $i = 1, \ldots, m - 1$ and $r_m > 0$ minimal (if $r_m = 0$, then the other $r_i$ must be integers, we recall). I claim that $w_1, \ldots, w_{m-1}, \lambda'$ must be a $\mathbb{Z}$-basis for $\mathcal{L}$. Indeed, if we pick any element $\eta \in \mathcal{L}$ and
write
\[ \eta = \sum_{i=1}^{m-1} a_i w_i + a_m v_m \]
with \( a_i \in \mathbb{R} \). Then by writing
\[ a_m = tr_m + z \]
with \( t \in \mathbb{Z} \) and \( 0 \leq z < r_m \) and subtracting
\[ \sum_{i=1}^{m-1} ([a_i - r_i t]) w_i + t \lambda' \]
from \( \eta \) we obtain an element of \( \mathcal{L} \) written as
\[ \sum_{i=1}^{m-1} ( (a_i - r_i t) - [a_i - r_i t] ) w_i + z v_m \]
with \( 0 \leq z < a_m \). Thus, we must have \( z = 0 \) and
\[ \eta - t \lambda' \in \mathcal{L}_O \]
and we are done. \( \square \)

Let’s define some notation now. For a finitely generated abelian group \( G \) we define \( \text{rk}(G) \) to be the free rank of \( G \). Let’s also define \( H \) to be the hyperplane \( x_1 + \ldots + x_{s+r} = 0 \) in \( \mathbb{R}^{s+r} \).

**Proposition 33.2.** \( \ell(\mathcal{O}_L^*) \) is a sublattice in \( H \).

**Proof.** Any bounded region in \( \mathbb{R}^{s+r} \) is contained in a set \( Y_C \) consisting of all \( (x_1, \ldots, x_{s+r}) \) with \( |x_i| \leq C \) for \( C \geq 0 \). For \( b \in \mathcal{O}_L \), the absolute value of the \( i \)-th coordinate of \( \ell(b) \) is less than or equal to \( C \) only if \( |\sigma_i(b)| \leq e^C \) for all \( i \). There are only finitely many such \( b \) by a Lemma from last time. \( \square \)

**Corollary 33.3.**
\[ \text{rk}(\mathcal{O}_L^*) \leq (r + s - 1) \]

**Proof.** Since the kernel of \( \ell \) is finite,
\[ \text{rk}(\mathcal{O}_L^*) = \text{rk}(\ell(\mathcal{O}_L^*)). \]
From the previous Proposition we know that \( \ell(\mathcal{O}_L^*) \) is sublattice in a vector space of dimension \( s + r - 1 \), so it must have \( \mathbb{Z} \)-rank at most \( s + r - 1 \). \( \square \)
We’re going to want use another embedding of $\mathcal{O}_L$ into an $\mathbb{R}$-vector space. This embedding, which we denote as $h^*$ is almost exactly like the embedding $h$ that we used earlier. It is

$$h^*(b) = (\sigma_1(b), \ldots, \sigma_r(b), \sigma_{r+1}(b), \ldots, \sigma_{r+s}(b)).$$

Note that is very similar to the embedding $h$ used earlier. In fact, we can choose the $\mathbb{R}$-basis $x_1, \ldots, x_r, y_1, z_1, \ldots, y_s, z_1, \ldots, z_s$, where $x_j$ is the element with $j$-th coordinate equal to 1 and all other coordinates equal to 0, $y_j$ to be the the element with $(r + j)$-th element equal to 1 and all other coordinates equal to 0, and $z_j$ to be the the element with $(r + j)$-th element equal to $i$ and all other coordinates equal to 0. Then $h$ is exactly the same with respect to its usual basis for $V$ as $h^*$ is with respect to the basis

$$x_1, \ldots, x_r, y_1, \ldots, y_s, z_1, \ldots, z_s.$$

If we give $\mathbb{R}^r \times \mathbb{C}^s$ the volume form associated to this basis, then

$$\text{Vol}(h^*(\mathcal{O}_L)) = \text{Vol}(h(\mathcal{O}_L)) = 2^{-s}\sqrt{\Delta(L/K)}.$$

In particular, $h^*(\mathcal{O}_L)$ is a full lattice in $\mathbb{R}^r \times \mathbb{C}^s$ (if it had $\mathbb{R}$-rank less than $n$, the volume would be 0).

The advantage of working with $h^*$ is that $\ell$ is that if we denote as $p_j$ projection onto the $j$-th coordinate (for $\mathbb{R}^r \times \mathbb{C}^s$). then

$$p_j(\ell(b)) = \log |p_j(h^*(b))|$$

for $1 \leq j \leq r$ and

$$p_j(\ell(b)) = 2\log |p_j(h^*(b))|$$

for $r + 1 \leq j \leq r + s$.

We have already established that $h^*(\mathcal{O}_L)$ is a lattice so we should be able to find elements in it with certain properties. The idea roughly is this: we want to find a family of units $u_i$ in $h^*(\mathcal{O}_L)$ for which we can control the $\pm$ sign of $\log |p_j(h^*(b))|$ for various $j$. We might hope that these units are linearly independent.

We will work with a region somewhat similar to the region we worked on when we were doing the finiteness of the class group. We define the region as follows. Let $(t)$ be an $(r+s)$-tuple of positive numbers indexed as $(t)_i$. We define

$$Z_{(t)} := \{(x_1, \ldots, x_{r+s}) \in \mathbb{R}^r \times \mathbb{C}^s \mid |x_i| \leq (t)_i, 1 \leq i \leq r$$

and $|x_i|^2 \leq (t)_i$ for $r + 1 \leq i \leq r + s\}$$
The region $Z(t)$ is just a cross product of regions in $\mathbb{R}$ and $\mathbb{C}$, specifically it is

$$[-(t)_1, (t)_1] \times \cdots \times [-t_r, (t)_r]$$

$$\times \{(x, y) \mid x^2 + y^2 \leq (t)_{r+1}^2\} \times \cdots \times \{(x, y) \mid x^2 + y^2 \leq (t)_{r+s}\}.$$

Thus,

$$\text{Vol}(Z(t)) = 2^r \pi^s t_1 \cdots t_{r+s}$$