Recall the definition of $Z(t)$ from last time...

Let $(t)$ be an $(r+s)$-tuple of positive numbers indexed as $(t)_i$. We define

$$Z(t) := \{(x_1, \ldots, x_{s+r}) \in \mathbb{R}^r \times \mathbb{C}^s \mid |x_i| \leq (t)_i, 1 \leq i \leq r$$

and $|x_i|^2 \leq (t)_i$ for $r + 1 \leq i \leq r + s\}.$

The region $Z(t)$ is just a cross product of regions in $\mathbb{R}$ and $\mathbb{C}$, specifically it is

$$[-(t)_1, (t)_1] \times \cdots \times [- (t)_r, (t)_r]$$

$$\times \{(x, y) \mid x^2 + y^2 \leq (t)_{r+1}\} \times \cdots \times \{(x, y) \mid x^2 + y^2 \leq (t)_{r+s}\}.$$

Thus,

$$\text{Vol}(Z(t)) = 2^r \pi^s t_1 \cdots t_{r+s}$$

And $Z(t)$ is convex and centrally symmetric. Now, let’s fix a constant $T$, for which

$$2^r \pi^s T^{r+s} > 2^n \text{Vol}(h^*(\mathcal{O}_L))$$

and let $(\gamma)$ be any $n$-tuple of numbers for which

$$\gamma_1 \cdots \gamma_{r+s} = 1.$$

Then

$$\text{Vol}(Z(T\gamma)) = 2^r \pi^s T^n > 2^n \text{Vol}(h^*(\mathcal{O}_L)),$$

so there exists a nonzero $b \in Z(T\gamma) \cap h^*(\mathcal{O}_L)$, by Minkowski’s lemma proven earlier. As said earlier, we want to control the signs of the logs of our units, so we will pick a particular $(\gamma)$ where $(\gamma_i) < 1$ for all but one $i$. Specifically, we pick a number $\epsilon$ and define

$$(\epsilon_i) = \begin{cases} 
\epsilon & : j \neq i \\
1/\epsilon^{r+s-1} & : j = i
\end{cases}$$

As above, we know that there is a nonzero element of $h^*(\mathcal{O}_L)$ in $Z(T\epsilon_i)$, call it $b_i$. The following Lemma is obvious. We state it to organize our exposition.

**Proposition 34.1.** Let $b_i \in Z(T\epsilon_i) \cap h^*(\mathcal{O}_L)$ with $b_i \neq 0$. Then

$$|N(b_i)| \leq T^{r+s}.$$
Proof. Recall of course that \( p_j(h^*(b)) = \sigma_j(b) \), so if \( h^*(b) \in Z_{(T\epsilon_i)} \), then \( |\sigma_j(b)| \leq (T\epsilon_i)_j \) for \( 1 \leq j \leq r \) and \( |\sigma_j(b)|^2 \leq (T\epsilon_i)_j \) for \( r + 1 \leq j \leq (s + r) \). Thus, for \( b_i \in Z_{(T\epsilon_i)} \), we have

\[
|N(b_i)| \leq \prod_{j=1}^{r} |\sigma_j(b)| \prod_{j=r+1}^{s+r} |\sigma_j(b)|^2 \leq \prod_{j=1}^{r+s} (T\epsilon_i)_j = T^{r+s}.
\]

\( \square \)

Unfortunately, the \( b_i \) are not units. However, we need only modify them slightly to obtain units. There are only finitely many nonzero principal ideals in \( \mathcal{O}_L \) with norm less than \( T^{r+s} \) (since there are finitely many ideals in \( \mathcal{O}_L \) of bounded norm). Let us number them as \( I_1, \ldots, I_N \), write \( I_k = \mathcal{O}_L a_k \), for \( a_k \in \mathcal{O}_L \) and pick \( \epsilon > 0 \) such that

\[
0 < \epsilon T < \min\{ |\sigma_i(a_k)|^{e_i}, i = 1, \ldots, r + s, k = 1, \ldots, N \},
\]

where \( e_i = 1 \) if \( \sigma_i \) is a real place and \( e_i = 2 \) is \( \sigma_i \) is complex place. Note that this \( \min \) cannot be zero because \( a_k \neq 0 \) and \( \sigma_i \) is injective. For each \( i = 1, \ldots, r + s \), let \( Z_{(T\epsilon_i)} \) and \( b_i \) be as in the Proposition above.

Since \( N(\mathcal{O}_L b_i) \leq T^{r+s} \), the ideal \( \mathcal{O}_L b_i \) is equal to some \( \mathcal{O}_L a_k(i) \). Let \( u_i = a_k(i)/b_i \). Then, \( u_i \) must be a unit since \( b_i \) divides \( a_k(i) \) and \( a_k(i) \) divides \( b_i \).

**Proposition 34.2.** Let \( u_i \) be as above. Then

1. \( \sum_{j=1}^{r} \log |\sigma_j(u_i)| + \sum_{j=r+1}^{r+s} 2 \log |\sigma_j(u_i)| = 0 \)
2. \( \log |\sigma_j(u_i)| < 0 \) for \( j \neq i \)
3. \( \log |\sigma_i(u_i)| > 0 \).

**Proof.** (1): This is easy since \( |N(u_i)| = 1 \), so

\[
0 = \log 1 = \log |N(u_i)| = \sum_{j=1}^{r} \log |\sigma_j(u_i)| + \sum_{j=r+1}^{r+s} 2 \log |\sigma_j(u_i)| = 0.
\]

(2): Recall that \( T\epsilon < |\sigma_j(a_{i(k)})| \), so

\[
\log |\sigma_j(u_i)^{e_i}| = \log |\sigma_j(b_i)^{e_i}/|\sigma_j(a_{i(k)})|^{e_i}| < \log \frac{T\epsilon}{|\sigma_j(a_{i(k)})|^{e_i}} < \log 1 = 0.
\]

Thus, \( \log |\sigma_j(u_i)| = \frac{1}{2} \log |\sigma_j(u_i)^{e_i}| < 0 \) as well.

(3): Follows immediately from (1) and (2) \( \square \)

**Proposition 34.3.** The elements \( \ell(u_i), i = 1, \ldots, r + s - 1 \) (note we don’t go up all the way to \( r + s \)) are linearly independent over \( \mathbb{R} \).
Proof. Let \( m_{ij} = \log |\sigma_j(u_i)| \) for \( 1 \leq i \leq r \) and \( m_{ij} = 2 \log |\sigma_j(u_i)| \) for \( r+1 \leq i \leq r+s-1 \). Since
\[
\sum_{j=1}^{r} \log |\sigma_j(u_i)| + \sum_{j=r+1}^{r+s} 2 \log |\sigma_j(u_i)| = 0,
\]
the \( \log |\sigma_{r+s}(u_j)| \) is determined by the other \( \log |\sigma_j(u_i)| \); that is why we only go up to \( r+s-1 \). To show that the \( \ell(u_i) \) are linearly independent, it will suffice to show that the matrix \([m_{ij}]\) is nonsingular. It follows from Proposition 34.2 that for any \( i \), we have
\[
\sum_{j=1}^{r+s-1} m_{ij} > 0.
\]
It also follows that \( m_{ij} < 0 \) for \( i \neq j \) and \( m_{jj} > 0 \) for any \( j \).

The embeddings of a fixed \( u_i \) gives us the \( i \)-th row of \([m_{ij}]\); it will be easier to show that the columns are linearly independent over \( \mathbb{R} \). Suppose that we have a set \( a_1, \ldots, a_{r+s-1} \) of real numbers, not all of which are zero. We can show that there is some \( i \) such that
\[
\sum_{j=1}^{r+s-1} a_j m_{ij} \neq 0.
\]
Indeed, let us pick \( i \) so that \( |a_i| \geq |a_j| \) for for all \( j \); we may assume that \( a_i > 0 \) since multiplying everything though by \(-1\) will not affect whether or not a sum is nonzero. Then we \( a_i \geq a_j \) for every \( j \) and (since \( m_{ij} < 0 \) for \( i \neq j \)) we have
\[
\sum_{j=1}^{r+s-1} a_j m_{ij} \geq a_i m_{ii} + \sum_{j \neq i} a_i m_{ij} \geq a_i \sum_{j=1}^{r+s-1} m_{ij} > 0
\]
and we are done. \( \square \)

Corollary 34.4. \( \ell(\mathcal{O}_L^*) \) is a full lattice in \( H \).

Proof. We have already seen that \( \ell(\mathcal{O}_L^*) \) is a lattice in \( H \). It is a full lattice since it generates a \( \mathbb{R} \)-vector space of dimension \( r+s-1 \), which must be equal to \( H \) (since \( \dim_{\mathbb{R}} H = r+s-1 \)). \( \square \)

Theorem 34.5 (Dirichlet Unit Theorem). Let \( \mu_L \) be the roots of unity in \( L \). There exist elements \( v_1, \ldots, v_{r+s-1} \in \mathcal{O}_L^* \) such that every unit \( u \in \mathcal{O}_L^* \) can be written uniquely as
\[
u v_1^{m_1} \cdots v_{r+s-1}^{m_{r+s-1}}
\]
for \( v \in \mu_L \) and \( m_i \in \mathbb{Z} \).
Proof. Let $v_1, \ldots, v_{r+s-1}$ have the property that $\ell(v_1), \ldots, \ell(v_{r+s-1})$ generate $\ell(O_L^*)$ as a $\mathbb{Z}$-module. Since $\ker \ell = \mu_L$, we know that every unit $u \in O_L^*$ can be written as $vz$, where $z$ is in the subgroup generated by the $v_1, \ldots, v_{r+s-1}$. The element $z$ is uniquely determined by $\ell(u)$ as
\[ v_1^{m_1} \cdots v_{r+s-1}^{m_{r+s-1}} \]
for some integers $m_i$. Then $v = zu^{-1}$ and is therefore also uniquely determined. □