There is one important difference between the $p$-adic absolute values and the ones coming from embedding $L$ in to $\mathbb{C}$, the so-called real absolute values. This difference lies in a stronger form of the triangle inequality satisfied by the $p$-adic absolute values. Recall that a valuation $v : K^* \rightarrow \mathbb{R}$ is a multiplicative map for which $v(x+y) \geq \min(v(x), v(y))$ for any $x, y \in K^*$. This last condition means that

$$|x+y| = e^{-(v(x)+v(y))} \leq e^{-\min(v(x), v(y))} \leq \max(|x|, |y|).$$

On the other hand, for the real valuation $|\cdot|$, we have, for example $|1+1| = 2 > \max(1, 1)$.

A valuation $v$ is called a discrete valuation if $v : K^* \rightarrow \mathbb{Z} \subseteq \mathbb{R}$ surjectively. By convention, we set $v(0) = \infty$.

**Definition 40.1.** If $|x+y| \leq \max(|x|, |y|)$ for every $x, y \in K$, then $|\cdot|$ is called an nonarchimedean valuation. Otherwise, it is called an archimedean valuation.

**Example 40.2.** Let $L = k(x)$ for $k$ any field. Since $B = k[x]$ is a PID, it is Dedekind. Thus, for any prime $P$ of $B$, the localization $B_P$ is a DVR. Hence, for any irreducible polynomial $P \in k[x]$, we have a discrete valuation $v_P$ on $L$, where $v_P(Q)$ is the highest power of $P$ dividing $Q$ (which is taken to be $\infty$ when $Q = 0$) for $Q \in B$ and $v_P(Q/R) = v_P(Q) - v_P(R)$ for $Q, R \in B$ and $R \neq 0$.

**The product formula.**

Suppose that we normalize the $p$-adic absolute values; that is, we set $\|x\|_p = p^{-v_p(x)}$. Then for any $x$, we have

$$\prod_P \|x\|_P = \frac{1}{p_1^{e_1} \cdots p_m^{e_m}}$$

where $x = \pm p_1^{e_1} \cdots p_m^{e_m}$. Let $\|x\|_\infty$ denote the usual absolute value $|x|$. Let

$$M_Q = \{ \text{primes } p \} \cup \infty.$$

Then

$$\prod_{v \in M_Q} \|x\|_P = 1.$$

This is called the product formula.

Similarly, working over $K[x]$, we call $P \in K[X]$ if $P$ is monic, irreducible, and has degree greater than 0. We let $\|x\|_P = e^{-v_P(x)(\deg P)}$. 

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Then letting \( \|x\|_\infty = e^{\deg x} \) (this measures the degree of the poll that \( x \) has at infinity), and letting
\[
M_{K[x]} = \{ \text{primes } P \in K[X] \} \cup \infty,
\]
we have
\[
\prod_{v \in M_{K[x]}} \|x\|_v = 1.
\]

Let \( K \) be a field with a discrete valuation \( v \) and let \( \hat{K} \) be the completion of \( K \) with respect to \( \cdot \) \( v \) where
\[
|x|_v = e^{-v(x)}.
\]
We define \( B_v \) to be the set of \( x \in \hat{K} \) with \( v(x) \geq 0 \) and let \( M_v \) be the maximal ideal in \( B_v \). We see below that \( B_v \) is indeed a DVR.

**Proposition 40.3.** With notation as above, \( v \) extends to a discrete valuation on \( \hat{K} \).

**Proof.** We take \( v(x) = -\log \lim |x_i| \) for \( x \neq 0 \) represented by \( (x_i)^\infty \).
To see that this actually gives an integer, write \( \lim |x_i| = C \) and if \( -\log C \) is not an integer, we can pick \( \epsilon \) so that \( C - e^{-m} > \epsilon \) for all integers \( m \). Then for any \( x_i \), we have \( |x_i| - \lim |x_i| > \epsilon \), which is impossible. Checking that \( v(x) \) is multiplicative and \( v(x + y) \leq \max(v(x), v(y)) \) is simple. \( \square \)