Math 531 Problem Set #2 Due 9/15/04

1. (a) Suppose that $\alpha$ is integral of degree 2 over $\mathbb{Z}$ and that there is a basis $m_1, m_2$ for $\mathbb{Z}[\alpha]$ over $\mathbb{Z}$ such that

\[
\begin{align*}
\alpha m_1 &= m_2 \\
\alpha m_2 &= -m_1 + m_2.
\end{align*}
\]

Find the quadratic integral equation satisfied by $\alpha$.

(b) Suppose that $\beta$ is integral of degree 2 over $\mathbb{Z}$ and that there is a basis $m_1, m_2$ for $\mathbb{Z}[\beta]$ over $\mathbb{Z}$ such that

\[
\begin{align*}
\beta m_1 &= m_2 \\
\beta m_2 &= 5m_1.
\end{align*}
\]

Find the quadratic integral equation satisfied by $\beta$.

2. Let $T$ be an $n \times n$ matrix with coefficients in a ring $A$. Show that there is a matrix $U$ for which $UT = (\det T)I$.

3. Let $A, B,$ and $C$ be rings with $A \subset B \subset C$. Show that if $B$ is integral over $A$ and $C$ is integral over $B$, then $C$ is integral over $A$.

4. (Ex. 4, p. 7) Let $d$ be a squarefree integer. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{d}]$ is

\[
\begin{align*}
\mathbb{Z}[\sqrt{d}] & \quad \text{if } d \equiv 2, 3 \pmod{4}, \\
\mathbb{Z} \left[\frac{1 + \sqrt{d}}{2}\right] & \quad \text{if } d \equiv 1 \pmod{4}.
\end{align*}
\]

5. (a) Let $\phi : A \to B$ be a mapping of rings. Show that for any prime ideal $\mathcal{P}$ in $B$, the ideal $\phi^{-1}(\mathcal{P})$ is a prime ideal in $A$.

(b) Give an example of a surjective ring homomorphism $\phi : A \to B$ for which there is a prime ideal $\mathcal{P}$ of $A$ such that $\phi(\mathcal{P})$ is not a prime ideal.

6. (a) Give an example of a mapping of rings $\phi : A \to B$ for which there is an ideal $I$ of $A$ such that $\phi(I)$ is not an ideal.

(b) Let $\phi : A \to B$ be a surjective mapping of rings. Show that for any ideal $I$ of $A$, the set $\phi(I)$ forms an ideal in $B$.

(c) Let $\phi : A \to B$ be any mapping of rings. Show that for any ideal $J$ of $B$, the set $\phi^{-1}(J)$ forms an ideal in $A$.

7. Let $A \subset B$ where $A$ and $B$ are domains and let $K$ be the field of fractions of $B$. Show that if $B$ is integrally closed over $A$ in $K$, then $B$ is integrally closed over itself in $K$.

8. (Ex. 4, p.3) Show that if $S$ is a multiplicative set not containing 0 in a Noetherian integral domain $\mathcal{R}$, then $S^{-1} \mathcal{R}$ is also a Noetherian integral domain.