1. Let \( R \) be an integral domain, let \( \mathcal{M} \) be a maximal ideal of \( R \), and let \( S \) be a multiplicative subset of \( R \) contained in \( R \setminus \mathcal{M} \). Show that
\[
R/\mathcal{M} \cong S^{-1}R/(S^{-1}R\mathcal{M}^n).
\]

2. Let \( \theta = \sqrt{3} \) and let \( R = \mathbb{Z} [\theta] \).
   (a) Write down a dual basis for the basis \( 1, \theta \) for \( \mathbb{Q}(\theta) \) over \( \mathbb{Q} \) with respect to the bilinear form \( (x, y) = \mathbb{T}_{\mathbb{Q}(\theta)/\mathbb{Q}}(xy) \).
   (b) Letting \( R^\dagger \) denote the \( \mathbb{Z} \)-module generated by the dual basis above, find the order of the abelian group \( R^\dagger / R \).

3. Let \( \theta = \frac{1 + \sqrt{5}}{2} \) and let \( R = \mathbb{Z} [\theta] \).
   (a) Write down a dual basis for the basis \( 1, \theta \) for \( \mathbb{Q}(\theta) \) over \( \mathbb{Q} \) with respect to the bilinear form \( (x, y) = \mathbb{T}_{\mathbb{Q}(\theta)/\mathbb{Q}}(xy) \).
   (b) Letting \( R^\dagger \) denote the \( \mathbb{Z} \)-module generated by the dual basis above, find the order of the abelian group \( R^\dagger / R \).

4. Let \( K \) be a field. We define the resultant \( \text{Res}(f, g) \) as follows. Write
\[
f(x) = b \prod_{i=1}^{m} (x - \alpha_i), \quad g(x) = c \prod_{j=1}^{n} (x - \beta_j),
\]
with \( b, c \in K^* \) and \( \alpha_i, \gamma_j \) in some algebraic closure of \( K \). Then
\[
\text{Res}(f, g) := b^n c^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j).
\]
   (a) Let \( h(x) = x^2 - 3 \). Calculate \( \text{Res}(h(x), h'(x)) \).
   (b) Let \( t(x) = x^2 - x - 1 \). Calculate \( \text{Res}(t(x), t'(x)) \).
   (c) Compare your answers to 2(b) and 3(b).

5. Let \( f \) and \( g \) be as above and suppose that \( g \) is nonconstant. Show that
\[
\text{Res}(f(x), g(x)) = b^n \prod_{i=1}^{m} g(\alpha_i).
\]

6. Let \( f \) be as above and assume that its leading coefficient \( b \) is 1 and that the degree of \( f \) is at least 2. Show that
\[
\text{Res}(f(x), f'(x)) = \prod_{1 \leq i, j \leq m, i \neq j} (\alpha_i - \alpha_j).
\]

7. Let \( K \subseteq L \) be a separable field extension with \( L = K(\theta) \) for an algebraic \( \theta \) with minimal monic polynomial \( f(x) \) over \( K \). Show that for \( a \in K \), we have \( N_{L/K}(a - \theta) = f(a) \).
   (b) Let \( \xi_p \) be a primitive \( p \)-th root of unity for a prime number \( p \). Show that \( N_{\mathbb{Q}(\xi_p)/\mathbb{Q}}(1 - \xi_p) = p \).