1. Let $A$ be a nonzero commutative ring that is not a field. Let $I$ be a nonzero proper ideal of $A$. Show that the covariant functor that takes the $A$-module $M$ to the $A$-module $M/IM$ and takes $A$-module homomorphisms $\phi : M \rightarrow N$ to maps $\phi_I : M/IM \rightarrow N/IN$ is always right-exact but is not always left-exact (this means that if $\phi$ is surjective then $\phi_I$ is surjective, but for some injective $\phi$, the map $\phi_I$ is not injective).

2. Let $G$ be an abelian group. We say that $M$ is a $G$-module if $M$ is an abelian group and $G$ acts on $M$ in such a way that $\sigma(a + b) = \sigma(a) + \sigma(b)$ for $\sigma \in G$ and $a, b \in M$. We say that $\phi : M \rightarrow N$ is a homomorphism of $G$-modules if $\phi$ is a group homomorphism and $\phi(\sigma(a)) = \sigma(\phi(a))$ for every $a \in M$. For a $G$-module $M$ we define $M^G$ to be the set of all $a \in M$ such that $\sigma(a) = a$ for every $\sigma \in G$. Let $F^G$ be the functor from the category of $G$-modules to the category of abelian groups that sends $M$ to $M^G$ and sends a $G$-module homomorphism $\phi : M \rightarrow N$ to $\phi^* : M^G \rightarrow N^G$. Show that $F^G$ is always left-exact and give an example of some $G$ for which $F^G$ is not right-exact.

3. Let $K$ be a field and let $A = K[X]$. Give an example of a torsion-free $A$-module which is not a free $A$-module.

4. Let $A$ be a principal ideal domain and let $M$ be a finitely generated $A$-module. Show that for any submodule $H$ of $M$, there exists a submodule $N$ of $M$ such that $N \cong M/H$.

5. Lang, Chapter XIV, Ex. 1.

6. Lang, Chapter XIV, Ex. 6.

7. Let $V$ be a finite dimensional vector space of dimension $n > 0$ over a field $K$. A bilinear form on $V$ is map $V \times V \rightarrow K$ denoted by $(a, b) \mapsto [a, b]$ such that for every $a, b, c, d, r \in K$, we have

$$[a, b] + [c, b] = [a + c, b]$$
$$[a, b] + [a, d] = [a, b + d]$$
$$[ra, b] = [a, rb] = r[a, b].$$

Let $e_1, \ldots, e_n$ be a basis for $V$ over $K$.

(a) Show that for any bilinear form $[,]$ there is matrix $M = [m_{ij}] \in M_{nn}(K)$ such that for any $a = \sum_{i=1}^{n} x_i e_i$ and $b = \sum_{i=1}^{n} y_i e_i$ in $V$, we have

$$[a, b] = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

(b) We say that $[,]$ is non degenerate if there does not exist any $x \in V \setminus \{0\}$ such that
\([x, y] = 0\) for every \(y \in V\) or such that \([y, x] = 0\) for every \(y \in V\). Show that \([\cdot, \cdot]\) is nondegenerate if and only if the matrix \(M\) in (a) has nonzero determinant.

8. Lang, Chapter IV, Ex. 1.

9. Lang, Chapter IV, Ex. 2.

10. Lang Chapter IV, Ex. 3. [Note: A field \(k\) is said to have characteristic 0 if \(m \cdot 1 \neq 0\) for any positive integer \(m\).]