p-ADIC LOGARITHMS FOR POLYNOMIAL DYNAMICS

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Abstract. We prove a dynamical version of the Mordell-Lang conjecture for subvarieties of the affine space $\mathbb{A}^g$ over a p-adic field, endowed with polynomial actions on each coordinate of $\mathbb{A}^g$. We use analytic methods similar to the ones employed by Skolem, Chabauty, and Coleman for studying diophantine equations.

1. Introduction

The Mordell-Lang conjecture was proved by Faltings [Fal94].

Theorem 1.1 (Faltings). *Let G be an abelian variety defined over the field of complex numbers $\mathbb{C}$. Let $X \subset G$ be a closed subvariety and $\Gamma \subset G(\mathbb{C})$ a finitely generated subgroup of $G(\mathbb{C})$. Then $X(\mathbb{C}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.*

In particular, Theorem 1.1 says that if a subvariety X of an abelian variety G does not contain a translate of a positive dimension algebraic subgroup of G, then X has a finite intersection with any finitely generated subgroup of $G(\mathbb{C})$. Theorem 1.1 has been generalized to semiabelian varieties G by Vojta (see [Voj96]) and to finite rank subgroups Γ of G by McQuillan (see [McQ95]). Recall that a semiabelian variety (over $\mathbb{C}$) is an extension of an abelian variety by a torus $(\mathbb{G}_m)^k$. The authors have proved the following dynamical version of Theorem 1.1 (see [GT]): if ϕ is an endomorphism of a semiabelian variety G defined over $\mathbb{C}$, and $V \subset G$ has no positive dimensional subvariety invariant under ϕ, then any orbit of ϕ has finite intersection with V. In case $G = \mathbb{G}_m^k$, this says that if an affine variety $V \subset \mathbb{G}_m^k$ contains no subvariety which is invariant under the map $(X_1, \dots, X_k) \mapsto (X_1^{e_1}, \dots, X_k^{e_k})$ (with $e_i \in \mathbb{N}$), then V has finite intersection with the orbit of any point of $\mathbb{A}^k$ under the above map on G. It is natural to ask whether a similar conclusion holds for any polynomial action on $\mathbb{A}^k$. In this spirit, we propose the following conjecture.

Conjecture 1.2. *Let $f_1, \dots, f_g$ be polynomials in $\mathbb{C}[X]$, let $\mathcal{P}$ be their action coordinatewise on $\mathbb{A}^g$, and let V be a subvariety of $\mathbb{A}^g$ that does not contain*

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a positive dimensional subvariety periodic under the $\mathcal{P}$-action. Then the $\mathcal{P}$-orbit of any point in $\mathbb{A}^g$ intersects V in at most finitely many points.

We also propose the following more general conjecture for the structure of the intersection of an affine variety with a polynomial orbit. We let $\mathcal{O}_{\mathcal{P}}((a_1,\dots,a_g))$ denote the $\mathcal{P}$-orbit of $(a_1,\dots,a_g)\in\mathbb{A}^g(\mathbb{C})$.

Conjecture 1.3. *With the above notation, each subvariety V of $\mathbb{A}^g$ defined over $\mathbb{C}$ intersects $\mathcal{O}_{\mathcal{P}}((a_1,\dots,a_g))$ in a finite union of orbits of the form $\mathcal{O}_{\mathcal{P}^N}(\mathcal{P}^\ell(a_1,\dots,a_g))$, for some non-negative integers N and ℓ.*

Conjecture 1.2 is an easy corollary of Conjecture 1.3. Indeed, if the intersection $V(\mathbb{C})\cap\mathcal{O}_{\mathcal{P}}((a_1,\dots,a_g))$ is infinite, then there exists an infinite orbit $\mathcal{O}_{\mathcal{P}^N}(\mathcal{P}^\ell(a_1,\dots,a_g))$ contained in V, and the Zariski closure of $\mathcal{O}_{\mathcal{P}^N}(\mathcal{P}^\ell(a_1,\dots,a_g))$ contains a positive dimensional subvariety of V invariant under $\mathcal{P}^N$. Note that Conjecture 1.3 says that if S is the set of non-negative integers n for which $\mathcal{P}^n(a_1,\dots,a_g)$ lies in V, then S is a finite union of translates of semigroups of $\mathbb{N}$.

Conjectures 1.2 and 1.3 fit into Zhang's far-reaching system of dynamical conjectures [Zha06]. Zhang's conjectures include dynamical analogues of the Manin-Mumford and Bogomolov conjectures for abelian varieties (now theorems of Raynaud [Ray83a, Ray83b], Ullmo [Ull98], and Zhang [Zha98]), as well as a conjecture about the Zariski density of orbits of points under fairly general maps from a projective variety to itself. The latter conjecture is related to our Conjecture 1.2, though neither conjecture contains the other.

Conjecture 1.3 is proved in [GT] when each $f_i\in\bar{\mathbb{Q}}[X]$, and $\deg(f_i)\le 1$. We also note that in [GTZ07], Conjecture 1.2 is proved in the special case where $g=2$ and V is a line in $\mathbb{A}^2$.

An analog of our Conjecture 1.3 for the additive group of positive characteristic under the action of an additive polynomial associated to a Drinfeld module has been previously studied (see [Den92], [Ghi05], [Ghi06] and [GT07]). However, Conjecture 1.2 seems more difficult, since the proofs over Drinfeld modules make use of the fact that the polynomials there are additive; in particular, this allows for the definition of a logarithm-like map that is defined on all of $\mathbb{G}_a$.

In the present paper we prove a first result towards Conjecture 1.2, one that is valid for polynomials defined over the p-adics. The idea behind the proof of our Theorem 2.2 can be explained quite simply. Assuming that an affine variety $V\subset\mathbb{G}_a^g$ has infinitely many points in common with an orbit $\mathcal{O}$ of a point which lies in a sufficiently small neighborhood of an attracting fixed point for $\mathcal{P}$, we can find then a positive dimensional subvariety of V which is invariant under $\mathcal{P}$. Indeed, applying the p-adic logarithmic map associated to $\mathcal{P}$ (see Proposition 3.3) to $\mathcal{O}$ yields a line in the vector space $\mathbb{C}_p^g$. Each polynomial f that vanishes on V, then gives rise to an analytic function F on this line (by composing with the p-adic exponential function associated to $\mathcal{P}$). Because we assumed there are infinitely many points in $V \cap \mathcal{O}$, the zeros of F must have an accumulation point on this line, which means that F vanishes identically on the line (by Lemma 4.1). The Zariski closure of this analytic line is a subvariety of V which is $\mathcal{P}$-invariant. The inspiration for this idea comes from the method employed by Chabauty in [Cha41] (and later refined by Coleman in [Col85]) to study rational points on curves in abelian varieties with low Mordell-Weil rank. Our technique also bears a resemblance to Skolem's method for treating diophantine equations (see [BS66, Chapter 4.6]).

We briefly sketch the plan of our paper. In Section 2 we set up the notation and state our main result (Theorem 2.2). Section 3 is devoted to proving several lemmas for p-adic logarithms associated to (one variable) polynomial maps on $\mathbb{C}_p$; these lemmas are used in the proof of Theorem 2.2. After that, in Section 4 we complete the proof of Theorem 2.2.

2. Notation

Let p be a prime number and let $\mathbb{C}_p$ be the completion of an algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. We let $|\cdot|$ be the absolute value on $\mathbb{C}_p$. For $\alpha \in \mathbb{C}_p$ and a real number $R > 0$, we let

$$B(\alpha; R) := \{z \in \mathbb{C}_p : |z - \alpha| < R\}.$$

An *isometry* between $B(\alpha_1; R)$ and $B(\alpha_2; R)$ is a map

$$\psi : B(\alpha_1; R) \longrightarrow B(\alpha_2; R)$$

such that for each $z \in B(\alpha_1; R)$, we have

$$|\psi(z) - \alpha_2| = |z - \alpha_1|.$$

For every polynomial $P \in \mathbb{C}_p[X]$, and for every $n \in \mathbb{N}$, we let P^n denote the n-th iterate of P; that is, we let

$$P^n := P \circ P \circ \cdots \circ P \ (n \text{ times}).$$

We call $\alpha \in \mathbb{C}_p$ a *preperiodic* point for P if there exist non-negative integers $n \neq m$ such that $P^n(\alpha) = P^m(\alpha)$.

Let $g \geq 2$, and let $P_1, \dots, P_g \in \mathbb{C}_p[X]$. We denote as $\mathcal{P}$ the action of $(P_1, \dots, P_g)$ on $\mathbb{A}^g$ coordinatewise. For each $(x_1, \dots, x_g) \in \mathbb{C}_p^g$, we define the *($\mathcal{P}$-) orbit* of $(x_1, \dots, x_g)$ be

$$\mathcal{O}_{\mathcal{P}}((x_1, \dots, x_g)) := \{\left(P_1^n(x_1), \dots, P_g^n(x_g)\right) : n \geq 0\}.$$

We will use the following classical definition from complex dynamics.

Definition 2.1. *Let P be a polynomial in $\mathbb{C}_p[z]$. We call $\alpha \in \mathbb{C}_p$ a* **fixed point** *of P if $P(\alpha) = \alpha$. We say that a fixed point α is an* **attracting** *fixed point for P if $0 < |P'(\alpha)| < 1$. If $P'(\alpha) = 0$ for a fixed point α, then we call α a* **superattracting** *fixed point. We call $P'(\alpha)$ the* **multiplier** *of α.*

The following is our main result.

Theorem 2.2. *With the above notation for $\mathcal{P}$, let $(\alpha_1,\dots,\alpha_g)\in\mathbb{C}_p^g$, and assume for each i that α_i is an attracting (but not superattracting) fixed point for P_i. In addition, suppose that the α_i have the same multiplier a_1. Let V be an affine subvariety of $\mathbb{A}^g$ defined over $\mathbb{C}_p$. Assume that V does not contain a positive dimensional subvariety invariant under the $\mathcal{P}$-action. Then there exists $R>0$ such that for any $(x_1,\dots,x_g)\in\mathbb{C}_p^g$ that satisfies $\max_{i=1}^g |x_i-\alpha_i|<R$, the intersection $V(\mathbb{C}_p)\cap\mathcal{O}_{\mathcal{P}}((x_1,\dots,x_g))$ is finite.*

3. p-adic logarithms

We will prove Theorem 2.2 after we develop a theory of p-adic logarithms associated to polynomials defined over $\mathbb{C}_p$. We begin with a variant of the classical Königs linearization of a polynomial P at an attracting (but not superattracting) fixed point α (see Theorem 8.2 in [Mil99], or Theorem 2.1 in [CG93, Chapter 2]). In both of those books, the result is proved over the complex numbers, and under the hypothesis that $0<|P'(\alpha)|<1$. By contrast, our result is over the p-adics and is less restrictive in that we only require $P'(\alpha)$ be neither 0 nor a root of unity. Our proof is also different than the proofs from the above mentioned books.

We note that we get a *convergent* power series for $\exp_P$ as long as $P'(\alpha)$ is neither 0, nor a root of unity. The reason roots of unity are a problem is that we divide out by $P'(\alpha)^n-1$ for various n when we are solving for the coefficients of $\exp_P$. In order to control the size of the coefficients, we will need a lemma on the size of $|P'(\alpha)^n-1|$. We begin with the following.

Lemma 3.1. *Let $\beta\in\mathbb{C}_p$ have the property that $|\beta-1|<|p|$. Then for all positive integers n, we have*

$$|\beta^n-1|=|\beta-1|\cdot|n|.$$

Proof. We use induction on the maximal power of p that divides n. If this power is zero, then we have

$$\begin{aligned}|\beta^n-1|&=|\beta-1|\cdot|1+\beta+\cdots+\beta^{n-1}|\\&=|\beta-1|\cdot\left|0+(\beta-1)+\cdots+(\beta^{n-1}-1)+n\right|\\&=|\beta-1|\cdot|n|,\end{aligned}$$

since $|\beta^j-1|<|n|=1$ for all j. To perform the inductive step, we let $n'=n/p$ and similarly obtain $|\beta^n-1|=|\beta^{n'}-1|\cdot|p|$, since $|(\beta^{n'})^j-1|<|p|$ for all j. Then, by the inductive hypothesis, we have

$$|\beta^n-1|=|(\beta^{n'})^p-1|=|\beta-1||n'|\cdot|p|=|\beta-1|\cdot|n|,$$

as desired. □

The following Lemma is an immediate consequence of Lemma 3.1. Schinzel ([Sch74, Lemma 3]) uses a similar lemma but only proves it over number fields.

Lemma 3.2. *Assume $b \in \mathbb{C}_p \setminus \{0\}$ is not a root of unity. Then there exists $0 < C \le 1$ such that for every $n > 1$, we have $|b^n - b| \ge C \cdot |n-1|$.*

We are now ready to prove the existence of our exponential map when the fixed point $\alpha = 0$.

Proposition 3.3. *Let $P(X) = \sum_{i=1}^r a_i X^i \in \mathbb{C}_p[X]$, where $a_1 \neq 0$ is not a root of unity. Then there exists a power series*

$$\exp_P(X) = X + c_2X^2 + \cdots + c_nX^n + \cdots \in \mathbb{C}_p[[X]]$$

such that

$$P(\exp_P(X)) = \exp_P(a_1X)$$

and $\exp_P$ has a positive radius of convergence.

In particular, Proposition 3.3 shows that for every $n \ge 1$, we have

$$P^n(\exp_P(X)) = \exp_P(a_1^n X), \tag{3.3.1}$$

as an identity of formal power series. Also, we obtain that there exists a *logarithmic* function $\log_P$, which equals the inverse $\exp_P^{-1}$ of the *exponential* function from Proposition 3.3. Moreover, since the radius of convergence for $\exp_P$ is positive, and because the coefficient of the linear term of $\exp_P$ is equal to 1, there exists $r_0 > 0$ such that both $\exp_P$ and $\log_P$ are analytic isometries on $B(0; r_0) \subset \mathbb{C}_p$ (see Proposition 3.4). Finally, using (3.3.1), we also derive

$$\log_P(P^n(X)) = a_1^n \log_P(X). \tag{3.3.2}$$

Proof of Proposition 3.3. We let $c_1 = 1$ and solve inductively for c_n with $n \ge 2$ by equating the coefficient of X^n in $P(\exp_P(X))$ with the coefficient of X^n in $\exp_P(a_1X)$ (it is clear that the coefficient of X in both power series must be equal to a_1). The coefficient of X^n in $P(\exp_P(X))$ must equal

$$a_1c_n + \sum_{i=2}^r a_i \cdot \sum_{\substack{j_1+\cdots+j_i=n \\ j_1,\ldots,j_i \ge 1}} \left(\prod_{\ell=1}^i c_{j_\ell} \right).$$

The coefficient of X^n in $\exp_P(a_1X)$ must equal $c_n a_1^n$. Thus, we may solve for c_n by letting

$$(a_1^n - a_1)c_n = \sum_{i=2}^r a_i \cdot \sum_{\substack{j_1+\cdots+j_i=n \\ j_1,\ldots,j_i \ge 1}} \left(\prod_{\ell=1}^i c_{j_\ell} \right). \tag{3.3.3}$$

Note that since a_1 is neither 0 nor a root of unity, we have $a_1^n - a_1 \neq 0$, so this equation does indeed have a solution for c_n in terms of a_1 and the c_j for which $j < n$. This shows the existence of the formal power series for the exponential function $\exp_P$.

We now prove that this formal power series has a positive radius of convergence in $\mathbb{C}_p$. Let $M := \min\{1, \frac{1}{\max_{i=2}^r |a_i|}\}$. Since a_1 is neither a root of unity nor equal to 0, Lemma 3.2 implies that there exists $0 < C \le 1$ such that for each $n \geq 2$, we have $|a_1^n - a_1| \geq C|n-1|$. We show by induction on n that $|c_n| \leq (C \cdot M)^{1-n} \cdot |(n-1)!|^{-1}$ for each $n \geq 1$.

For $n = 1$, we know that $c_1 = 1$, so the desired inequality holds.

Now, let $n \geq 2$. Assume that $|c_i| \leq (C \cdot M)^{1-i}|(i-1)!|^{-1}$ for each $i < n$; we will show that $|c_n| \leq (C \cdot M)^{1-n}|(n-1)!|^{-1}$. Using (3.3.3) and the ultrametric inequality, we conclude that

$$C|n-1| \cdot |c_n| \leq |a_1^n - a_1| \cdot |c_n| \leq \left(\max_{i=2}^{r} |a_i|\right) \cdot \left(\max_{\substack{j_1+\cdots+j_i=n \\ j_1,\ldots,j_i \geq 1}} \prod_{\ell=1}^{i} |c_{j_\ell}|\right). \tag{3.3.4}$$

Since $|a_i| \leq \frac{1}{M}$, the induction hypothesis (applied to the various c_{j_ℓ}) above thus yields

(3.3.5)

$$C|n-1| \cdot |c_n| \leq \left(\frac{1}{M}\right) \cdot \left(\max_{i=2}^{r} (C \cdot M)^{i-n}\right) \cdot \left(\max_{\substack{j_1+\cdots+j_i=n \\ i \geq 2}} \prod_{\ell=1}^{i} |(j_\ell - 1)!|^{-1}\right).$$

Both C and M are in $(0, 1]$, so $\max_{i=2}^{r} (C \cdot M)^{i-n} = (C \cdot M)^{2-n}$. For each $i, j_1, \ldots, j_i$ such that $j_1 + \cdots + j_i = n$, we have

$$\frac{(n-i)!}{(j_1 - 1)! \cdots (j_i - 1)!} \in \mathbb{Z}.$$

This gives $\prod_{\ell=1}^{i} |(j_i - 1)!|^{-1} \leq |(n-i)!|^{-1} \leq |(n-2)!|^{-1}$ and, thus, (3.3.5) implies that

$$C|n-1| \cdot |c_n| \leq C^{2-n} \cdot M^{1-n} \cdot |(n-2)!|^{-1}, \tag{3.3.6}$$

which completes the inductive step.

Since $|(n-1)!|^{-1} \leq |p|^{-\frac{n-1}{p-1}} \leq |p|^{-n}$, we conclude that $|c_n| \leq (CM|p|)^{-n}$. Hence $\exp_P$ is convergent in the ball $B(0; CM|p|)$. □

The following result is classical in non-archimedean analysis. For the sake of completeness, we provide its proof.

Proposition 3.4. *Let $f(X) = X + c_2X^2 + \cdots + c_nX^n + \cdots \in \mathbb{C}_p[[X]]$ be a power series convergent on a ball $B(0; r)$ of positive radius. Then there exists $r_0 \in (0, r]$ such that for each $z \in B(0; r_0)$, we have $|f(z)| = |z|$. Moreover, f admits an analytic inverse function f^{-1} on $B(0; r_0)$.*

Proof. Let $0 < r_1 < r$. Then $\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} < \frac{1}{r_1}$. Therefore, there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, we have $|c_n|^{\frac{1}{n}} < \frac{1}{r_1}$. In particular, there exists $K > 0$ such that for *all* $n \geq 2$, we have

$$|c_n| < \frac{K}{r_1^n}. \tag{3.4.1}$$

We can find $0 < r_0 < r_1$ such that for all $n \geq 2$, we have

$$r_0^{n-1} < \frac{r_1^n}{K}. \tag{3.4.2}$$

We claim that for each $z \in B(0; r_0)$, we have $|f(z)| = |z|$. Indeed, for each $n \geq 2$, using (3.4.1), (3.4.2) and that $|z| < r_0$, we have

$$|c_n z^n| < \frac{|z|^n \cdot K}{r_1^n} < |z|.$$

Hence $|f(z)| = |z|$, as desired. Moreover, for each $w \in B(0; r_0)$, there exists $z \in B(0; r_0)$ such that $f(z) = w$ (see the first slope of the Newton polygon for the polynomial $f(X) - w$). Thus $f : B(0; r_0) \to B(0; r_0)$ is a bijection.

Since f is a unit in the ring of formal power series, there exists a formal power series f^{-1} which is the inverse of f. Furthermore, the fact that f is one-to-one and onto $B(0; r_0)$ means that f^{-1} is well-defined and analytic on $B(0; r_0)$ (because f is analytic on $B(0; r_0)$). Furthermore, $|f^{-1}(z)| = |z|$, for each $z \in B(0; r_0)$ since $|f(z)| = |z|$ for each $z \in B(0; r_0)$. □

The following result is a nonarchimedean version of a classical result in dynamics. It follows in a similar manner as Proposition 3.4 from considering $P(X)$ as a polynomial in $(X - \alpha)$ and applying the ultrametric inequality for nonarchimedean absolute values.

Fact 3.5. *Let $P \in \mathbb{C}_p[X]$ and let $\alpha \in \mathbb{C}_p$ be a fixed point for P. Assume that $0 < |P'(\alpha)| < 1$ (i.e. α is an attracting fixed point for P). Then there exists $R > 0$ such that for each $z \in B(\alpha; R) \setminus \{0\}$, we have*

$$|P(z) - \alpha| = |P'(\alpha)| \cdot |z - \alpha| < |z - \alpha|. \tag{3.5.1}$$

Thus, there exists $R > 0$ such that for every $z \in B(\alpha; R)$, we have

$$\lim_{n \to \infty} P^n(z) = \alpha.$$

Using Proposition 3.3, we construct a p-adic logarithmic function in a neighborhood of any attracting fixed point of a polynomial P.

Proposition 3.6. *Let $P(X) \in \mathbb{C}_p[X]$, and suppose that $\alpha \in \mathbb{C}_p$ is a fixed point of P. Assume that $0 < |P'(\alpha)| < 1$. Then there exists $R > 0$, and there exist convergent power series $\exp_P : B(0; R) \to B(\alpha; R)$ and $\log_P : B(\alpha; R) \to B(0; R)$ (which are inverses of each other), such that $P(\exp_P(X)) = \exp_P(b_1 X)$ and $\log_P(P(X)) = b_1 \log_P(X)$, where $b_1 := P'(\alpha)$.*

Proof. The proof is a simple change of coordinates argument followed by the application of Proposition 3.3. Let $G(X) := P(X + \alpha) - \alpha$. Then $G(0) = 0$ and $G'(0) = P'(\alpha) = b_1$. Note that b_1 is neither 0 nor a root of unity. By Propositions 3.3 and 3.4, we have analytic exponential and logarithmic functions for G in a neighborhood of 0; we denote these as $\exp_G$ and $\log_G$. Recall that $\log_G$ and $\exp_G$ are isometries on $B(0; R)$ for sufficiently small R. At the expense of shrinking R, we may assume that P maps $B(\alpha; R)$ into itself (see Fact 3.5), not necessarily onto. Then $\exp_P(X) := \exp_G(X) + \alpha$ and $\log_P(X) := \log_G(X - \alpha)$ are inverses of each other, and for any $X \in B(\alpha; R)$

(see Proposition 3.3), we have

$$\begin{aligned}\log_P(P(X)) &= \log_G(P(X)-\alpha)\\ &= \log_G(G(X-\alpha))\\ &= b_1\log_G(X-\alpha)\\ &= b_1\log_P(X),\end{aligned} \tag{3.6.1}$$

and similarly, we have $P(\exp_P(X)) = \exp_P(b_1X)$ for $X \in B(0;R)$. □

4. Proof of our main result

We are now almost ready to prove Theorem 2.2, which is the main result of this paper. The proof makes use of the fact that the zeros of any analytic function are isolated, unless the function is identically zero. The following lemma is standard (see [GT07, Lemma 3.4], for example).

Lemma 4.1. *Let $F(z) = \sum_{i=0}^{\infty} a_i(z-\alpha)^i$ be a power series with coefficients in $\mathbb{C}_p$ that is convergent in an open disc B of positive radius around the point $z = \alpha$. Suppose that F is not the zero function. Then the zeros of F in B are isolated.*

We now begin the proof of Theorem 2.2.

Proof. Let $R > 0$ satisfy the hypothesis of Fact 3.5 for each P_i. Let $x_i \in B(\alpha_i;R)$ for each $i \in \{1,\dots,g\}$. If each $x_i = \alpha_i$, then $\mathcal{O}_{\mathcal{P}}((x_1,\dots,x_g)) = (\alpha_1,\dots,\alpha_g)$, and so, the conclusion of Theorem 2.2 is immediate. Hence, we may assume that

$$0 < \max_{i=1}^{g} |x_i - \alpha_i| < R. \tag{4.1.1}$$

Assume there exists an infinite sequence of non-negative integers $\{n_k\}_{k\geq 0}$ such that $\mathcal{P}^{n_k}(x_1,\dots,x_g) \in V$. We will show that V must contain a positive dimensional subvariety V_0 such that $\mathcal{P}(V_0) = V_0$. By Fact 3.5, we have

$$P_i^{n_k}(x_i) \to \alpha_i \text{ for each } i \in \{1,\dots,g\}. \tag{4.1.2}$$

After replacing $\{n_k\}_k$ by a subsequence and R by a smaller positive number, we may assume that $\log_{P_i}$ is well-defined at $P_i^{n_k}(x_i)$ for each $i \in \{1,\dots,g\}$ and $k \geq 0$, by (4.1.2) and Proposition 3.6. Similarly, after shrinking R further, we may also assume that the analytic maps $\log_{P_i}$ and $\exp_{P_i}$ (defined as in Proposition 3.6) are isometries (see also Proposition 3.4).

Because not all $x_i = \alpha_i$, we may assume that

$$\left|\log_{P_1}\left(P_1^{n_0}(x_1)\right)\right| = \max_{i=1}^{g}\left|\log_{P_i}\left(P_i^{n_0}(x_i)\right)\right| > 0, \tag{4.1.3}$$

using the fact that $\log_{P_i}$ is an isometry between $B(\alpha_i,R)$ and $B(0;R)$.

We will need to use the following claim.

Claim 4.2. *For each $i \in \{2,\dots,g\}$, the fraction $\lambda_i := \frac{\log_{P_i}\left(P_i^{n_k}(x_i)\right)}{\log_{P_1}\left(P_1^{n_k}(x_1)\right)}$ is independent of $k \geq 0$.*

Proof of Claim 4.2. First of all, it follows immediately from (4.1.3) and Fact 3.5 that the denominator of λ_i is not zero. Then, using (3.6.1), we see that

$$\log_{P_1}(P_1^{n_k}(x_1)) = a_1^{n_k-n_0}\cdot\log_{P_1}\left(P_1^{n_0}(x_1)\right) \tag{4.2.1}$$

Similarly, we obtain $\log_{P_i}(P_i^{n_k}(x_i)) = a_1^{n_k-n_0}\cdot\log_{P_i}\left(P_i^{n_0}(x_i)\right)$. This proves Claim 4.2. □

For each polynomial $f\in\mathbb{C}_p[X_1,\dots,X_g]$ in the vanishing ideal of V, we construct the following power series

$$F(u) := f\left(u, \exp_{P_2}\left(\lambda_2\cdot\log_{P_1}(u)\right),\dots,\exp_{P_g}\left(\lambda_g\cdot\log_{P_1}(u)\right)\right).$$

Using (4.1.3) and Claim 4.2, we conclude that $|\lambda_i|\le 1$ for each $i\in\{2,\dots,g\}$. Therefore, for each i we have that $\lambda_i\cdot\log_{P_1}(u)\in B(0;R)$ if $u\in B(\alpha_1;R)$. Hence F is analytic on $B(\alpha_1;R)$ (we also use the fact that f is a polynomial).

Using Claim 4.2, we conclude that for each $k\ge 0$, we have

$$\lambda_i\cdot\log_{P_1}\left(P_1^{n_k}(x_1)\right) = \log_{P_i}\left(P_i^{n_k}(x_i)\right)\ \text{ and so,} \tag{4.2.2}$$

$$F(P_1^{n_k}(x_1)) = f\left(P_1^{n_k}(x_1), P_2^{n_k}(x_2),\dots,P_g^{n_k}(x_g)\right) = 0,$$

where in the last equality we used that $\left(P_1^{n_k}(x_1),\dots,P_g^{n_k}(x_g)\right)\in V$. But $\lim_{k\to\infty}P_1^{n_k}(x_1)=\alpha_1$ (see (4.1.2)). Because the zeros of F have an accumulation point inside its domain of convergence, we conclude that $F=0$ (see Lemma 4.1). Thus, we have

$$f\left(u, \exp_{P_2}\left(\lambda_2\log_{P_1}(u)\right),\dots,\exp_{P_g}\left(\lambda_g\log_{P_g}(u)\right)\right) = 0$$

for all polynomials f in the vanishing ideal of V, and for each $u\in B(\alpha_1;R)$. In particular, each f vanishes on the set

$$\mathcal{S} = \left\{\left(P_1^n(x_1),\dots,P_g^n(x_g)\right)\ :\ n\ge n_0\right\}$$

Let W_0 be the Zariski closure of $\mathcal{S}$ and let V_0 be the union of the positive dimensional components of W_0 (note that V_0 is nonempty since the set $\mathcal{S}$ is infinite). Then V_0 is a subvariety of V, because any polynomial f vanishes on V_0 whenever f vanishes on V. Since $\mathcal{P}(\mathcal{S})$ and $\mathcal{S}$ differ by at most one point (namely, the point $\mathcal{P}^{n_0}(x_1,\dots,x_g)$), we see that $\mathcal{P}(V_0)=V_0$.

Thus, if V contains infinitely many points in the $\mathcal{P}$-orbit of $(x_1,\dots,x_g)$, then V must contain a $\mathcal{P}$-invariant subvariety of positive dimension. □

The following result is a corollary of our Theorem 2.2

Corollary 4.3. *Let $g\ge 2$, and for each $i\in\{1,\dots,g\}$ we let*

$$P_i(X) = a_1X + \sum_{j=2}^{d_i} a_{i,j}X^j \in \mathbb{C}_p[X].$$

Assume that $0<|a_1|<1$ and that for each $i\in\{1,\dots,g\}$, and for each $j\ge 2$, we have $|a_{i,j}|\le 1$.

Let $V \subset \mathbb{A}^g$ be an affine subvariety defined over $\mathbb{C}_p$ which contains no positive dimensional subvariety invariant under the $\mathcal{P}$-action. Then for each g-tuple of p-adic integers $(\alpha_1, \dots, \alpha_g) \in \mathbb{C}_p$ the $\mathcal{P}$-orbit of $(\alpha_1, \dots, \alpha_g)$ intersects $V(\mathbb{C}_p)$ in at most finitely many points.

The proof of Corollary 4.3 is immediate after we notice that our hypothesis on the coefficients of the polynomials P_i and on the α_i guarantee us that for each $i \in \{1, \dots, g\}$, we have $\lim_{n\to\infty} P_i^n(\alpha_i) = 0$. Furthermore, 0 is an attracting fixed point for each P_i; therefore, Theorem 2.2 shows that V intersects the $\mathcal{P}$-orbit of $(\alpha_1, \dots, \alpha_g)$ in at most finitely many points.

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