The purpose of this homework is to see what approximation has to do with solutions to polynomial equations in two variables.

**Theorem 0.1** (Thue’s Theorem). Let $\alpha$ be any algebraic number in $\mathbb{C}$ and let $\epsilon > 0$ and let $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$. We call $d$ the **degree** of $\alpha$. Then there are finitely many $x/y \in \mathbb{Q}$ such that

$$|\alpha - x/y| < \frac{1}{|y|(d/2+1)+\epsilon}.$$ 

Now, we introduce a polynomial...

Let $F(x, y)$ be a homogeneous polynomial of degree $d$ with coefficients in $\mathbb{Z}$. This means that we have

$$F(x, y) = \sum_{i=0}^{d} a_i x^i y^{d-i},$$

where $a_i \in \mathbb{Z}$. Suppose that $F$ factors as

$$F(x, y) = \gamma \sum_{i=1}^{d} (x - \alpha_i y),$$

where $\gamma$ and all of the $\alpha_i$ are in $\overline{\mathbb{Q}}$.

In each of the following exercises, $F(x, y)$ is as defined above.

1. Suppose that $d \geq 3$ and that all of the $\alpha_i$ in equation (1) are distinct. Let $m$ be any nonzero integer. Use Thue’s theorem to show that there are finitely many integer pairs $(x, y)$ for which

$$F(x, y) = m.$$

2. Give an example of a polynomial $F$ as above with degree $d \geq 3$ for which there is a nonzero integer $m$ such that there are infinitely many integer pairs $(x, y)$ for which

$$F(x, y) = m.$$

3. Give an example of a polynomial $F$ of degree $d = 2$ such that

   (a) the $\alpha_i$ in (1) are distinct; and

   (b) there is a nonzero integer $m$ such that there are infinitely many integer pairs $(x, y)$ for which

$$F(x, y) = m.$$