Zeros of Partial Sums of the Riemann Zeta-Function

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Upstate Number Theory Conference 2012

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There have been numerical studies by R. Spira and, more recently, P. Borwein *et al.*.

Zeros of $F_{211}(s)$

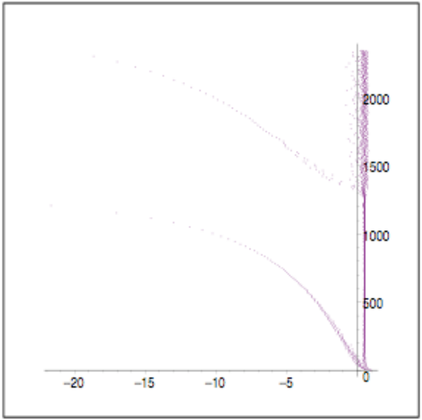

Figure: Zeros of $F_{211}(s)$ from P. Borwein et al.

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$$N_X(\sigma, T) = \sum_{\substack{0 \le \gamma_X \le T \\ \beta_X \ge \sigma}} 1.$$

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Here we are mostly concerned with the latter.

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$$\sigma > 1 + \frac{c \log \log X}{\log X}.$$

- (Montgomery & Vaughan) If X is sufficiently large, $F_X(s)$ has no zeros in

$$\sigma \geq 1 + \left(\frac{4}{\pi} - 1\right)\frac{\log \log X}{\log X}.$$

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$$\left| N_X(T) - \frac{T}{2\pi} \log [X] \right| < \frac{X}{2}.$$

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Here $[X]$ denotes the greatest integer less than or equal to X.

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If $T^{1/2} \ll X = o(T)$*, the* X *on the right-hand side is replaced by* T/X*.*

Idea of the proof: As for $\zeta(s)$: mollify $F_X(s)$ and apply Littlewood's lemma.

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for almost all zeros of ρ_X with $0 \leq \gamma_X \leq T$.

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for all zeros of ρ_X with $X^{1/2} \le \gamma_X \le T$.

Idea of the proof: On RH

$$\zeta(s) = F_X(s) + O\left(X^{1/2-\sigma} \exp\left(\frac{A \log t}{\log\log t}\right)\right)$$

for $9 \le X \le t^2$, and $1/2 \le \sigma \le 2$.

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$$\sum_{0 \leq \gamma_X \leq T} (\beta_X + U) = U \frac{T}{2\pi} \log X + O(UX) + O(T).$$

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$$2\pi \sum_{0 \leq \gamma_X \leq T} (\beta_X + U) = \int_0^T \log |F_X(-U + it)| dt + \cdots .$$

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- What proportion of the zeros of $F_X(s)$ have $\beta_X \ge 1/2$?