Understanding Dyson's Lemma

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(work in progress)

Abstract. In 1989, I proved a Dyson lemma for products of two smooth projective curves of arbitrary genus. In 1995, M. Nakamaye extended this to a result for a product of an arbitrary number of smooth projective curves of arbitrary genus, in a formulation involving an additional "perturbation divisor." In 1998, he also found an example in which a hoped-for Dyson lemma is false without such a perturbation divisor. This talk will present some recent work suggesting that it may be possible to eliminate the perturbation divisor by using a different definition of "volume" at the points under consideration.

Vague Definitions and History

Let $0 \neq P \in \mathbb{C}[x_1, x_2]$ be of degree d_1 in x_1 and d_2 in x_2 ($d_1 \gg d_2$), and let $Q_1, \dots, Q_s$ be points in $\mathbb{C}^2$ with distinct x_1 coordinates and distinct x_2 coordinates. Then Dyson's lemma says that

$$\sum_{i=1}^{s} \mathrm{Vol}_{P,d_1,d_2}(Q_i) \leq 1 + O(d_2/d_1) \; .$$

History:

Theorem (Roth). *Let $\alpha \in \overline{\mathbb{Q}}$, let $\epsilon > 0$, and let $C \in \mathbb{R}$. Then there are only finitely many $p/q \in \mathbb{Q}$ ($p, q \in \mathbb{Z}$, $\gcd(p,q) = 1$) such that*

$$\left| \frac{p}{q} - \alpha \right| \leq \frac{C}{|q|^{2+\epsilon}} \; .$$

1909	Thue	$\frac{d}{2} + 1 + \epsilon$
1921	Siegel	$\min\left\{ \frac{d}{s+1} + s : 0 \leq s < d \right\} + \epsilon$
1947	Dyson	$\sqrt{2d} + \epsilon$
1952	Gel'fond	$\sqrt{2d} + \epsilon?$
1955	Roth	$2 + \epsilon$

Detailed Description of Dyson's Lemma

Let $C_1,\dots,C_n$ be smooth projective curves over $\mathbb{C}$, let Y be an effective divisor on $C_1\times\dots\times C_n$, and let $d_i=(Y\,.\,\widetilde{C}_i)$ for all i, where $\widetilde{C}_i$ is a fiber of the map $C_1\times\dots\times C_n\to\prod_{j\ne i}C_j$. Assume that $d_i>0$ for all i.

Definition. For $P\in C_1\times\dots\times C_n$ define the index of Y at P relative to $\mathbf{d}=(d_1,\dots,d_n)$ as

$$t_{\mathbf{d},Y}(P)=\min\left\{\frac{i_1}{d_1}+\dots+\frac{i_n}{d_n}:\left(\frac{\partial}{\partial z_1}\right)^{i_1}\cdots\left(\frac{\partial}{\partial z_n}\right)^{i_n}f(P)\ne 0\right\},$$

where f is a local defining equation for Y at P and z_i are local coordinates on C_i.

We also define $\operatorname{Vol}(t)$ as

$$\operatorname{Vol}(t)=\text{volume of }\left\{(x_1,\dots,x_n)\in[0,1]^n:\sum x_i\le t\right\}.$$

Question. *Given $C_1,\dots,C_n,Y,d_1,\dots,d_n$ as above, and points $P_1,\dots,P_s\in\prod C_i$ lying in distinct fibers over C_i for all i, can one show that*

$$\sum_{i=1}^{s}\operatorname{Vol}(t_{\mathbf{d},Y}(P_i))\le\frac{1}{d_1\cdots d_n}\cdot\frac{(Y^n)}{n!}+O\left(\max\left\{\frac{d_i}{d_j}:i>j\right\}\right)$$

with the constant in $O(\cdot)$ depending only on $g(C_1),\dots,g(C_n)$, n, s?

The intuition behind this is that generally

$$h^0\left(\prod C_i,Y\right)\approx\frac{(Y^n)}{n!}$$

(if Y is ample), and $d_1\cdots d_n\cdot\operatorname{Vol}(t_{\mathbf{d},Y}(P_i))$ is the approximate number of linear conditions one would use to (naively) achieve the given index at P_i. Thus, the inequality becomes best possible in the limit as $\max\{d_i/d_j\}\to 0$.

More History

n	C_i	s	
2	$\mathbb{P}^1$	any	Dyson 1947 (some differences)
2	$\mathbb{P}^1$	any	Viola 1985
any	$\mathbb{P}^1$	any	Esnault-Viehweg 1984; Roth proof
2	any	any	V. 1989; new proof of Mordell
any	any	0	V. 1990 (unpublished)
any	any	any	Nakamaye 1995 "perturbation divisor"
counterexample			Nakamaye 1998

Proofs

When $n = 1$ (simple but instructive):

$$\sum \frac{\deg_{P_i}(Y)}{d_1} \le \frac{\deg Y}{d_1} \,.$$

No $O(\cdot)$ term

When $n = 2$ (discussion):

At one of the P_i, you can draw a Newton polygon for a defining equation for Y [on board].

If you work harder, you can get [on board]:

Why is the region cut off?

(a). Can't have ∞ on the LHS

(b). You get the rest "for free," so they shouldn't count.

Proposal for Vol when $n = 3$

Let $\mathrm{Vol}_{\mathscr{O}(Y),\mathbf{d}}$ be the volume of the set

$$\begin{aligned}\{(x,y,z) \in [0,\infty)^3 : \; & x \le 1,\, y \le 1,\, z \le 1,\\ & x+y \le t_{12},\, x+z \le t_{13},\, y+z \le t_{23},\\ & x+y+z \le t\}\end{aligned}$$

where t_{12} satisfies

$$\mathrm{Vol}_{Y|_{F_3},(d_1,d_2)}(t_{12}) = \frac{(Y^2 \,.\, F_3)}{2d_1 d_2}$$

and t_{13}, t_{23} are defined similarly; d_1, d_2, d_3 are as defined earlier; and F_i is a fiber of $C_1 \times C_2 \times C_3 \to C_i$.

For $n > 3$: you can see a pattern.

Why hasn't this come up before???

(a). It has ($n = 1$).

(b). When $n = 2$: no change

(c). When $n > 2$ and $C_i = \mathbb{P}^1$ for all i (say $n = 3$), $\mathscr{O}(Y) \cong \mathscr{O}(d_1, d_2, d_3)$, $(Y^2 \,.\, F_3) = 2d_1 d_2$, so $\mathrm{Vol}(t_{12}) = 1$, giving $t_{12} = 2$, etc.

Also, this definition addresses Nakamaye's counterexample.

It also fits in with the principle of not giving credit for things that are free, *including when you apply Dyson's lemma to the faces of the cube*.

Current Status

Proved when $n = 3$, $s = 1$ ($n \le 2$ already done).

Sketch of proof when $n = 2$, $s = 0$. First consider the special case when Y contains no components that are fibers of $C_1 \times C_2 \to C_1$ or $C_1 \times C_2 \to C_2$. If Z is an irreducible component of Y, then

$$(Z^2 + Z \,.\, K_{C_1 \times C_2}) = 2p_a(Z) - 2 \ge 2p_g(Z) - 2 \ge \deg(Z \to C_2)(2g(C_2) - 2)\,,$$

and therefore

$$(Z^2) \ge -(2g(C_1) - 2)(Z \,.\, (\{\text{pt.}\} \times C_2))\,.$$

Writing $Y = \sum e_k Z_k$, we then have

$$\begin{aligned}(Y^2) &\ge -\max\{e_k\}\max\{2g(C_1) - 2, 0\}(Y \,.\, (\{\text{pt.}\} \times C_2)) \\ &\ge -d_2^2 \max\{2g(C_1) - 2, 0\}\,.\end{aligned}$$

If Y contains fiber components, then the inequality is still true (and may be stronger).

Now divide by $2d_1d_2$. □

Sketch of proof when $n = 2$, $s = 1$. Again start with the case when Y contains no fiber components.

Take covers C_1', C_2' of C_1 and C_2, ramified only above the coordinates of $P = P_1$, and unramified elsewhere (unless $C_i = \mathbb{P}^1$, in which case you allow ramification above a second point). Moreover, we require that the ramification indices at all points over the coordinate of P all be the same, and occur in such a ratio such that the index of Y at P is some multiple of the straight multiplicity of the pull-back Y' at each point above P. Let X be the blowing-up of $C_1' \times C_2'$ at all points over P. Apply the above argument to the divisor Y'' obtained by subtracting suitable multiples of the exceptional divisors from Y', so that Y'' is not supported along any exceptional divisor. This gives

$$(Y^2) - t(P)^2 \ge -d_2^2 \max\{2g(C_1) - 2 + 1, 0\}\,.$$

Adding back in the fibers not passing through P again only makes things better, but things are more complicated when Y contains fibers that pass through P. Write

$$Y = Y_0 + aF_1 + bF_2\,,$$

where F_i is the fiber of $C_1 \times C_2 \to C_i$ passing through P. Then

$$\begin{aligned}(Y^2) &= (Y_0)^2 + 2a(Y_0 \,.\, F_1) + 2b(Y_0 \,.\, F_2) + 2ab \\ &= (Y_0)^2 + 2a(d_2 - b) + 2b(d_1 - a) + 2ab\end{aligned}$$

and dividing by 2 then gives the area of the region [draw].

Note that the region contains the region indicated by $\operatorname{Vol}(t)$.

□

[**Caution:** You only get the area of a smaller region when $s > 2$.]

[The proof when $s > 1$ is too messy to give here.]

Sketch of Proofs when $n = 3$, $s \le 1$

Sketch of proof when $n = 3$, $s = 0$. If Z_k is an irreducible component of Y, then looking at $(Z_k)^2$ is not good enough, nor is positivity of the relative dualizing sheaf useful in this case. So, instead, you prove that

$$Y + (d_2 + d_3)\pi_1^* K_1 + d_3 \pi_2^* K_2$$

is nef, where K_i is the pull-back of the canonical divisor on C_i (or the trivial sheaf if $C_i \cong \mathbb{P}^1$) and then you get

$$((Y + (d_2 + d_3)\pi_1^* K_1 + d_3\pi_2^* K_2)^3) \ge 0\,.$$

Actually, you can do a little better:

$$(Y \,.\, (Y + (d_2 + d_3)\pi_1^* K_1 + d_3\pi_2^* K_2)^2) \ge 0\,.$$ □

Sketch of proof when $n = 3$, $s = 1$. The same changes carry over: you do a covering construction to turn the index downstairs into the straight multiplicity upstairs, fibers F_i of $C_1 \times C_2 \times C_3 \to C_i$ not passing through P can be added back in without problem, and you write

$$Y = Y_0 + aF_1 + bF_2 + cF_3$$

as before, to get

$$\begin{aligned}Y^3 = Y_0^3 &+ 3aY_0^2F_1 + 3bY_0^2F_2 + 3cY_0^2F_3 \\ &+ 6ab(d_3 - c) + 6ac(d_2 - b) + 6bc(d_1 - a) + 6abc\end{aligned}$$

and apply

$$\begin{aligned}(Y_0^3) &\ge (t - a/d_1 - b/d_2 - c/d_3)^3 \\ 3aY_0^2F_1 &\ge 6a\operatorname{Vol}_{d_2,d_3}(t_{23})\,,\end{aligned}$$

etc.

Again, need to compare the volume of the implicit solid with $\operatorname{Vol}(t)$ proposed earlier. □