Problem Set #4
Math 453 – Differentiable Manifolds
Assignment: Chapter 5 #1
Chapter 6 #1, 2, 4

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February 13, 2013
Exercise 5.1

Let $A$ and $B$ be two points not on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$. For any two positive real numbers $c, d$, define

$$I_A(-c, d) = (-c, 0) \cup \{A\} \cup (0, d)$$

and similarly for $I_B(-c, d)$, with $B$ instead of $A$. Define a topology on $S$ as follows: on $(\mathbb{R} - \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis of neighborhoods at $A$ is the set $\{I_A(-c, d) : c, d > 0\}$; similarly, a basis of neighborhoods at $B$ is $\{I_B(-c, d) : c, d > 0\}$.

(a) Prove that the map $h : I_A(-c, d) \to (-c, d)$ defined by

$$h(x) = \begin{cases} x & : x \neq A \\ 0 & : x = A \end{cases}$$

is a homeomorphism.

(b) Show that $S$ is locally Euclidean and second countable, but not Hausdorff.

Solution.

(a) The map is clearly continuous from a standard $\varepsilon - \delta$ proof; likewise, it is bijective. Thus, the only thing to prove is that the inverse function is continuous, or equivalently, that $h$ is an open map. This is clear, however, since $f$ is injective. Let $U$ be an open subset of $S$. If $A \notin S$, then $f(U) = U$, which is open. If $A \in U$, then $U$ contains an 'interval' around $A$ with some other open sets, in particular $U = V \cup (-c, 0) \cup \{A\} \cup (0, d)$. Then $f(U) = f(V) \cup f((-c, 0)) \cup f(A) \cup f((0, d)) = f(V) \cup (-c, d)$, which is open.

(b) Since $h$ is a homeomorphism to an open interval, it is easily seen to be locally Euclidean, and is second countable as we can take open intervals with rational endpoints as a basis (and this works for sets including $A$). However, it is not Hausdorff, because we must consider $B$. The open sets containing $A$ are of the form $(-a, 0) \cup \{A\} \cup (0, b)$ and for $B$ we have $(-c, 0) \cup \{B\} \cup (0, d)$ for $a, b, c, d > 0$. Thus we have $(0, c) \cap (0, d) \neq \emptyset$, and therefore $A$ cannot be separated from $B$. Equivalently, one can argue $\lim_{n \to \infty} 1/n$ converges to $A$ and $B$, therefore the space is not Hausdorff since the limit is not unique.

Q.E.D.
Exercise 6.1

Let \( \mathbb{R} \) be the real line with the differentiable structure given by the maximal atlas of the chart \((\mathbb{R}, \varphi = \mathbb{R} : \mathbb{R} \to \mathbb{R})\), and let \( \mathbb{R}' \) be the real line with the differentiable structure given by the maximal atlas of the chart \((\mathbb{R}, \psi : \mathbb{R} \to \mathbb{R})\) where \( \psi(x) = x^{1/3} \).

(a) Show that these two differentiable structure are distinct.

(b) Show that there is a diffeomorphism between \( \mathbb{R} \) and \( \mathbb{R}' \).

Solution.

(a) Suppose the differential structure on the manifolds is the same so that the charts are compatible. Then \( \psi \circ \phi^{-1} \) should be \( C^\infty \), however, we know this is not the case since \( \psi \circ \phi^{-1}(x) = x^{1/3} \) is not \( C^\infty \) at \( x = 0 \).

(b) Define \( F : \mathbb{R} \to \mathbb{R}' \) by \( F(x) = x^3 \). Then \( \psi \circ F \circ \varphi^{-1}(x) = x \), that is, it is just the identity function, which is clearly \( C^\infty \). As it is the identity, the inverse function is the identity also, therefore this \( C^\infty \) function is, in fact, a diffeomorphism.

Q.E.D.
Exercise 6.2

Let $M$ and $N$ be manifolds and let $q_0$ be a point in $N$. Prove that the inclusion map $i_{q_0} : M \to M \times N$, $i_{q_0}(p) = (p, q_0)$, is $C^\infty$.

Solution.

This is immediate from Exercise 6.18. Since $i_{q_0} = (\iota_M, q_0) : M \to M \times N$, the identity map is $C^\infty$ and a constant map is $C^\infty$.

Q.E.D.
Exercise 6.4

Find all points in $\mathbb{R}^3$ in a neighborhood of which the functions $x$, $x^2 + y^2 + z^2 - 1$, $z$ can serve as a local coordinate system.

Solution.

Define $F(x, y, z) = (x, x^2 + y^2 + z^2 - 1, z)$. We know this will be precisely when the Jacobian determinant is nonzero:

$$\det \begin{vmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ 0 & 0 & 1 \end{vmatrix} \neq 0.$$  

Calculating the determinant, we get $2y$, thus it is precisely when $2y \neq 0$, or $y \neq 0$. Thus, $F$ can serve as a coordinate system at any point $p$ not on the $xz$-plane.

Q.E.D.