Problem Set #5
Math 453 – Differentiable Manifolds
Assignment: Chapter 7 #1, 5, 7, 8, 9, 11

Clayton J. Lungstrum
February 20, 2013
Exercise 7.1

Let $f : X \to Y$ be a map of sets, and let $B \subseteq Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. Therefore, if $f$ is surjective, then $f(f^{-1}(B)) = B$.

Solution.

Let $y \in f(f^{-1}(B))$. Then there is an $x \in f^{-1}(B)$ such that $f(x) = y$. By definition, this means $f(x) = y \in B$. Since $x \in X$, then we have $f(x) \in B \cap f(X)$, as desired. The reverse containment follows analogously.

Suppose $f$ is surjective; then $f(X) = Y$, so for any $B \subseteq Y$, $B \cap Y = B$. Therefore, $f(f^{-1}(B)) = B$, as was to be shown.

Q.E.D.
**Exercise 7.3**

*Deduce Theorem 7.7 from Corollary 7.8.*

**Solution.**

We want to show that a topological space $S$ is Hausdorff if and only if the diagonal $\Delta$ in $S \times S$ is closed implies that for an open equivalence relation $\sim$, the quotient space $S/\sim$ is Hausdorff if and only if the graph $R$ of $\sim$ is closed in $S \times S$.

First, suppose that $S/\sim$ is Hausdorff. Let $[x] \neq [y]$ be elements of $S/\sim$. Then there exist disjoint open subsets $U$ and $V$ of $S/\sim$ such that $[x] \in U$ and $[y] \in V$. Then $\pi^{-1}(U)$ is open by the continuity of $\pi$, and similarly for $\pi^{-1}(V)$. Also note that they are disjoint, hence $(x, y) \in \pi^{-1}(U) \times \pi^{-1}(V) \subseteq S \times S$ is an open set disjoint from $R$. It follows that since $x$ and $y$ were arbitrary, that $R$ is closed.

Conversely, suppose $R$ is closed. Then, for a point $(x, y) \notin R$, there is an open set, say $U$, containing it that does not intersect $R$. This implies there are open sets $V$ and $W$ such that $V \times W \subseteq U$ and the point is in $V \times W$. Thus, we get that $x \in V$ and $y \in W$. Now, the openness of $\pi$ implies that $\pi(x) \in \pi(V)$ and $\pi(y) \in \pi(W)$ where $\pi(V)$ and $\pi(W)$ are open disjoint sets, hence $S/\sim$ is Hausdorff.

Q.E.D.
Exercise 7.5

Suppose a right action of a topological group \( G \) on topological space \( S \) is continuous; this simply means that the map \( S \times G \to S \) describing the action is continuous. Define two points \( x, y \) of \( S \) to be equivalent if they are in the same orbit; i.e., there is an element \( g \in G \) such that \( y = xg \). Let \( S/G \) be the quotient space; it is called the orbit space of the action. Prove that the projection map \( \pi : S \to S/G \) is an open map. (This problem generalizes Proposition 7.14, in which \( G = \mathbb{R}^\times \) and \( S = \mathbb{R}^{n+1} \setminus \{0\} \). Because \( \mathbb{R}^\times \) is commutative, a left \( \mathbb{R}^\times \)-action becomes a right \( \mathbb{R}^\times \)-action if scalar multiplication is written on the right.)

Solution.

Let \( U \) be an open subset of \( S \). For each \( g \in G \), since right multiplication by \( g \) is a homeomorphism \( S \to S \), the set \( Ug \) is open. But \( \pi^{-1}(\pi(U)) = \bigcup_{g \in G} Ug \), which is a union of open sets, hence is open. By the definition of the quotient topology, \( \pi(U) \) is open.

Q.E.D.
Exercise 7.7

(a) Let \(\{(U_\alpha, \varphi_\alpha)\}_{\alpha=1}^2\) be the atlas of the circle \(S^1\) in Example 5.7, and let \(\varphi_\alpha\) be the map \(\varphi_\alpha\) followed by the projection \(\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}\). On \(U_1 \cap U_2 = A \cup B\), since \(\varphi_1\) and \(\varphi_2\) differ by an integer multiple of \(2\pi\), \(\varphi_1 = \varphi_2\). Therefore, \(\varphi_1\) and \(\varphi_2\) piece together to give a well-defined map \(\varphi: S^1 \to \mathbb{R}/2\pi\mathbb{Z}\). Prove that \(\varphi\) is \(C^\infty\).

(b) The complex exponential \(\mathbb{R} \to S^1, t \mapsto e^{it}\), is constant on each orbit of the action of \(2\pi\mathbb{Z}\) on \(\mathbb{R}\). Therefore, there is an induced map \(F: \mathbb{R}/2\pi\mathbb{Z} \to S^1, F([t]) = e^{it}\). Prove that \(F\) is \(C^\infty\).

(c) Prove that \(F: \mathbb{R}/2\pi\mathbb{Z} \to S^1\) is a diffeomorphism.

Solution.

Q.E.D.
Exercise 7.8

Exercise 7.9

Show that the real projective space $\mathbb{R}P^n$ is compact.

Solution.

By Exercise 11, we know it is homeomorphic to $S^n/\sim$, which is the continuous image of a compact set, hence $\mathbb{R}P^n$ is compact.

Q.E.D.
Exercise 7.11

For $\mathbf{x} = (x^1, \ldots, x^n) \in \mathbb{R}^n$, let $\|\mathbf{x}\| = \sqrt{\sum_i (x^i)^2}$ be the modulus of $\mathbf{x}$. Prove that the map $f : \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$f(x) = \frac{x}{\|x\|}$$

induces a homeomorphism $\overline{f} : \mathbb{R}^P \to S^n/\sim$.

Solution.

Define $\overline{f} : \mathbb{R}^P \to S^n/\sim$ by

$$\overline{f}([x]) = \left[ \frac{x}{\|x\|} \right].$$

It’s clear that this map is well-defined as multiplying it by any nonzero scalar would give the positive or negative of the equivalence class, but since we’re in a linear subspace, it’s the same. Now, by Proposition 7.1, the map $\overline{f}$ is continuous.

Now we can define $g : S^n \to \mathbb{R}^{n+1} - \{0\}$ by $g(x) = x$ and we see that it induces a map $\overline{g} : S^n \to \mathbb{R}^P$ which, by the argument above, is well-defined and continuous. Moreover,

$$\overline{g} \circ \overline{f}([x]) = [x]$$

and

$$\overline{f} \circ \overline{g}([x]) = [x].$$

Thus, we have that $\overline{f}$ and $\overline{g}$ are inverses to each other.

Q.E.D.