Problem Set #7
Math 453 – Differentiable Manifolds
Assignment: Chapter 9 #1, 2, 3, 5

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Exercise 9.1

Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by

\[
f(x, y) = x^3 - 6xy + y^2.
\]

Find all values \( c \in \mathbb{R} \) for which the level set \( f^{-1}(c) \) is a regular submanifold of \( \mathbb{R}^2 \).

Solution.

The only values \( c \in \mathbb{R} \) which have the property that \( f^{-1}(c) \) is not a regular submanifold are \( c = 0 \) as \( (0, 0) \) is a critical point and \( f(0, 0) = 0 \) and \( c = -108 \) as \( (6, 18) \) is a critical point and \( f(6, 18) = -108 \).

Q.E.D.
Exercise 9.2

Let \( x, y, z, w \) be the standard coordinates on \( \mathbb{R}^4 \). Is the solution set of \( x^5 + y^5 + z^5 + w^5 = 1 \) in \( \mathbb{R}^4 \) a smooth manifold? Explain why or why not. (Assume that the subset is given the subspace topology.)

Solution.

Define

\[
S = \{(x, y, z, w) \in \mathbb{R}^4 : x^5 + y^5 + z^5 + w^5 = 1 \}.
\]

Let \( g : \mathbb{R}^4 \to \mathbb{R} \) be defined as \( g(x, y, z, w) = x^5 + y^5 + z^5 + w^5 - 1 \) so that \( S = g^{-1}(1) \). Calculating the partial derivatives, the only critical point of \( g \) is \((0, 0, 0, 0)\), which is not in \( S \). Thus, 1 is a regular value of \( g \), and by the regular level set theorem, \( S \) is a regular submanifold of \( \mathbb{R}^4 \) of dimension 3. So \( S \) is a manifold.

Q.E.D.
Exercise 9.3

Is the solution set of the system of equations

\[ x^3 + y^3 + z^3 = 1, \quad z = xy, \]

in \( \mathbb{R}^3 \) a smooth manifold? Prove your answer.

Solution.

It is a regular submanifold since we can define \( F : \mathbb{R}^3 \to \mathbb{R}^2 \) by

\[
(u, v) = F(x, y, z) = (x^3 + y^3 + z^3, xy - z).
\]

We can compute the Jacobian matrix to be

\[
\begin{bmatrix}
3x^2 & 3y^2 & 3z^2 \\
y & x & -1
\end{bmatrix}
\]

Solving for when the \( 2 \times 2 \) minors are zero yields the equations \( x^3 - y^3 = 0 \), \( x^2 + yz^2 = 0 \), and \( y^2 + xz^2 = 0 \). Solving the system, we see that \( x = y \) from the first equation, and so from either of the second two equations, we see that \( x^2 = -xz^2 \). Now, if \( x = 0 \), then \( x = y = z = 0 \), but since \((0, 0, 0)\) is not part of the solution set, the regular level set theorem tells us it is a regular submanifold. If \( x \neq 0 \), then the fact that in our solution set, \( z = xy = x^2 \), we have (by dividing the equation by \(-x\)) \(-x = x^4\), which occurs if and only if \( x = 0 \) or \( x = -1 \). In the first case, we already have seen that this gives us a regular submanifold, so suppose \( x = -1 \). This implies \( y = -1 \), hence \( z = 1 \), but \((-1, -1, 1)\) is clearly not in the solution set by the equation \( x^3 + y^3 + z^3 = 1 \). Therefore, we have that the solution set is indeed a regular submanifold.

Q.E.D.
Exercise 9.5

Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\Gamma(f) = \{(x, y, f(x, y)) \in \mathbb{R}^3\},$$

is a regular submanifold of $\mathbb{R}^3$.

Solution.

We will be more general and prove it for $f : \mathbb{R}^m \to \mathbb{R}^n$. As such, define $\varphi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by $\varphi(x, y) = (x, y - f(x))$. Now we see that $\varphi(x, y) = (x, 0)$ if and only if $(x, y) \in \Gamma(f)$ (that is, $f(x) = y$). Clearly the function is injective, and we have

$$D\varphi = \begin{pmatrix} I & 0 \\ -Df & I \end{pmatrix}$$

which is nonsingular, hence $\varphi$ is a diffeomorphism. This means $\varphi$ is a chart for the graph of $f$, making it a regular submanifold.

Q.E.D.