Exercise 8.2

Prove the converse of Hölder’s inequality for \( p = 1 \) and \( p = \infty \). Show also that for real-valued \( f \not\in L^p(E) \), there exists a function \( g \in L^{p'}(E) \), \( \frac{1}{p} + \frac{1}{p'} = 1 \), such that \( fg \not\in L^1(E) \).

Solution.

Consider the case where \( p = 1 \) so that \( p' = \infty \). Then, by Hölder’s inequality, we certainly have the left-hand side majorizes the right-hand side, and we have equality if we let \( g = \text{sgn}(f) \). For then we have

\[
\int_E fg = \int_E |f| = \|f\|_1.
\]

By definition, \( \|g\|_\infty \leq 1 \), hence it is clear that for \( p = 1 \), there exists a \( g \in L^{p'}(E) \) such that

\[
\|f\|_p = \sup_{\|g\|_\infty \leq 1} \int_E fg.
\]

Alternatively, consider the case where \( p = \infty \). Then we have three cases. If \( \|f\|_\infty = 0 \), we have \( f = 0 \) a.e. which implies \( fg = 0 \) a.e., i.e. \( \int_E fg = 0 \) for all \( g \). Now, if \( \|f\|_\infty \) is positive and finite, we may assume that \( \|f\|_\infty = 1 \) without loss of generality. Then define the set \( E_n = \{ x \in E : |f(x)| > 1 - \frac{1}{n} \text{ and } |x| < n \} \) and note that \( 0 < |E_n| < \infty \) for each integer \( n \). Define

\[
g_n(x) = \begin{cases} 
\frac{1}{|E_n|} & : x \in E_n \\
0 & : x \not\in E_n
\end{cases}.
\]

Clearly \( \int_{E_n} g_n = 1 \). Now observe that

\[
\int_E |f| g_n = \int_{E_n} |f| g \geq \left(1 - \frac{1}{n}\right) \int_{E_n} g_n = 1 - \frac{1}{n}.
\]

Thus, taking the supremum over all such \( g \) with \( \|g\|_1 \leq 1 \), we have the desired equality. When \( \|f\|_\infty = \infty \), apply the same argument just used on the set \( F_n = \{ x \in E : |f(x)| > n \} \). This gives us a lower bound of \( n \), and taking the supremum over all positive integers yields the equality.

To show that for \( f \not\in L^p(E) \), there exists a \( g \in L^{p'}(E) \) such that \( fg \not\in L^1(E) \), consider the following. Without loss of generality, we can assume all functions are nonnegative. Now, suppose there is a sequence \( \{g_k\}_{k=1}^\infty \subseteq L^{p'}(E) \) with \( \|g_k\|_{p'} = 1 \) and

\[
\int_E f g_k > 3^k.
\]

Set

\[
g = \sum_{k=1}^\infty 2^{-k} g_k
\]

and observe that, by Minkowski’s inequality, \( \|g\|_{p'} \leq 1 \). Note that

\[
\int_E fg = \int_E f \sum_{k=1}^\infty 2^{-k} g_k > \int_E \sum_{k=1}^\infty \left(\frac{3}{2}\right)^k = \infty.
\]
Thus, \( g \in L^{p'}(E) \) but \( fg \notin L^1(E) \).

Hence, we have reduced the problem to showing that such a sequence exists. First note that if \( f = \infty \) on any set \( A \) of positive measure, then we can simply take

\[
g = \begin{cases} 
\frac{1}{|B|^{1/p'}} & : x \in B \\
0 & : x \notin B
\end{cases},
\]

where \( B \subseteq A \) has positive, finite measure. Thus \( g \) has the desired properties. This implies we can assume \( f \) is finite a.e., and in particular, for any positive real number \( c \), we can find a set \( F \) with finite measure such that \( \int_F f = c \). With this in mind, we can find a nested sequence of sets \( \{E_k\}_{k=1}^{\infty} \), each with finite measure, such that \( \bigcup_{k=1}^{\infty} E_k = E \) and \( \int_{E_k} f > 3^k \).

Now, take

\[
g_k = \begin{cases} 
\frac{1}{|E_k|^{1/p'}} & : x \in E_k \\
0 & : x \notin E_k
\end{cases}.
\]

By construction, \( \|g_k\|_{p'} = 1 \), and it is easy to check that \( \int fg_k \to \infty \) as \( k \to \infty \). Hence, we have the existence of the sequence, and we’ve demonstrated the existence of such functions \( g \).

\[\text{Q.E.D.}\]
Exercise 8.3

Prove Theorems (8.12) and (8.13). Show that Minkowski’s inequality for series fails when \( p < 1 \).

Solution.

Let us recall Theorem 8.12 and Theorem 8.13. Theorem 8.12, i.e., Hölder’s inequality for series, states the following:

Suppose that \( 1 \leq p \leq \infty \), \( \frac{1}{p} + \frac{1}{p'} = 1 \), \( a = \{a_k\}_{k=1}^{\infty} \), \( b = \{b_k\}_{k=1}^{\infty} \), and \( ab = \{a_kb_k\}_{k=1}^{\infty} \). Then

\[
\sum_{k=1}^{\infty} |a_kb_k| \leq \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{\infty} |b_k|^{p'} \right)^{\frac{1}{p'}} \quad \text{when } 1 < p < \infty
\]

\[
\sum_{k=1}^{\infty} |a_kb_k| \leq \left( \sup_{k \in \mathbb{N}} |a_k| \right) \left( \sum_{k=1}^{\infty} |b_k| \right) \quad \text{when } p = 1, \infty.
\]

The result is clear as the proof for the integral version goes through. For argument’s sake, we present it here. The second inequality is trivial since

\[
\sum_{k=1}^{\infty} |a_kb_k| \leq \sum_{k=1}^{\infty} \left( \sup_{k \in \mathbb{N}} |a_k| \right) |b_k| = \left( \sup_{k \in \mathbb{N}} |a_k| \right) \sum_{k=1}^{\infty} |b_k|.
\]

By symmetry of argument, this proves the inequality for the cases \( p = 1, \infty \). Now suppose \( 1 < p < \infty \). Then by Young’s inequality, we have

\[
\sum_{k=1}^{\infty} |a_kb_k| \leq \sum_{k=1}^{\infty} \left( \frac{|a_k|^p}{p} + \frac{|b_k|^{p'}}{p'} \right) = \frac{1}{p} \sum_{k=1}^{\infty} |a_k|^p + \frac{1}{p'} \sum_{k=1}^{\infty} |b_k|^{p'} = \frac{1}{p} \|a\|_p^p + \frac{1}{p'} \|b\|_{p'}^{p'} = \frac{1}{p} + \frac{1}{p'} = 1,
\]

where we have assumed \( \|a\|_p = \|b\|_{p'} = 1 \). Observe this is enough as we can set \( A = \frac{a}{\|a\|_p} \) and \( B = \frac{b}{\|b\|_{p'}} \) and check that \( \|A\|_p = 1 = \|B\|_{p'} \). Thus, Hölder’s inequality for series holds as stated for \( 1 \leq p \leq \infty \).

Theorem 8.13, i.e., Minkowski’s inequality for series, states the following:

Suppose that \( 1 \leq p \leq \infty \), \( a = \{a_k\}_{k=1}^{\infty} \), \( b = \{b_k\}_{k=1}^{\infty} \), and \( a + b = \{a_k + b_k\}_{k=1}^{\infty} \). Then

\[
\left( \sum_{k=1}^{\infty} |a_k + b_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{\infty} |b_k|^p \right)^{\frac{1}{p}} \quad \text{when } 1 \leq p < \infty
\]

\[
\sup_{k \in \mathbb{N}} |a_k + b_k| \leq \sup_{k \in \mathbb{N}} |a_k| + \sup_{k \in \mathbb{N}} |b_k| \quad \text{when } p = \infty.
\]
Again, the proof for the integral version goes through, but for argument’s sake, we’ll present it here. Observe that case $p = 1$ is just the standard triangle inequality for real numbers, so that case is finished. When $p = \infty$, note that by the triangle inequality, we always have $|a_k + b_k| \leq |a_k| + |b_k|$. By the definition of supremum, we must have $|a_k| \leq \sup_{k \in \mathbb{N}} |a_k|$ and $|b_k| \leq \sup_{k \in \mathbb{N}} |b_k|$. Thus

$$|a_k + b_k| \leq \sup_{k \in \mathbb{N}} |a_k| + \sup_{k \in \mathbb{N}} |b_k|.$$ 

Since the right-hand side is an upper bound, it must be at least as great as the least upper bound, in particular,

$$\sup_{k \in \mathbb{N}} |a_k + b_k| \leq \sup_{k \in \mathbb{N}} |a_k| + \sup_{k \in \mathbb{N}} |b_k|,$$

as desired. Now, suppose $1 < p < \infty$. Then,

$$\|a + b\|_p^p = \sum_{k=1}^{\infty} |a_k + b_k|^{p-1} |a_k + b_k|$$

$$\leq \sum_{k=1}^{\infty} |a_k + b_k|^{p-1} |a_k| + \sum_{k=1}^{\infty} |a_k + b_k|^{p-1} |b_k|$$

$$\leq \left( \sum_{k=1}^{\infty} |a_k + b_k|^p \right)^{\frac{p-1}{p}} \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{\frac{1}{p}}$$

$$+ \left( \sum_{k=1}^{\infty} |a_k + b_k|^p \right)^{\frac{p-1}{p}} \left( \sum_{k=1}^{\infty} |b_k|^p \right)^{\frac{1}{p}}$$

$$= \|a + b\|_p^{p-1} \|a\|_p + \|a + b\|_p^{p-1} \|b\|_p.$$

Now, to show the inequality cannot be improved upon, simply consider the sequences given by

$$a_k = \begin{cases} 1 & : k = 1 \\ 0 & : k > 1 \end{cases} \quad \text{and} \quad b_k = \begin{cases} 1 & : k = 2 \\ 0 & : k \neq 2 \end{cases}.$$ 

It is not hard to see Minkowski’s inequality is not true for any $0 < p < 1$ for these sequences, for observe

$$\left( \sum_{k=1}^{\infty} |a_k + b_k|^p \right)^{\frac{1}{p}} = 2^{\frac{1}{p}}.$$ 

Since $0 < p < 1$, we know $1 < \frac{1}{p} < \infty$, hence

$$2^{\frac{1}{p}} > 2 = 1 + 1 = \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{\infty} |b_k|^p \right)^{\frac{1}{p}}.$$ 

Q.E.D.
Exercise 8.6

Prove the following generalization of Hölder’s inequality. If \( \sum_{i=1}^{k} \frac{1}{r_i} = 1 \) with \( p_i, r \geq 1 \), then
\[
\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.
\]

Solution.

We’ll proceed by induction by first showing the inequality to be true when \( k = 2 \). To this end, consider
\[
\|fg\|_r^r = \int |fg|^r \leq \left( \int |f|^{r \cdot \frac{p_1}{r}} \right)^{\frac{r}{p_1}} \left( \int |g|^{r \cdot \frac{p_2}{r}} \right)^{\frac{r}{p_2}} = \|f\|_{p_1} \|g\|_{p_2},
\]
where
\[
\frac{1}{p_1} + \frac{1}{p_2} = 1,
\]
or equivalently,
\[
\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r}.
\]
Thus, the generalization holds when \( k = 2 \). Assume the inequality is true for some \( n \in \mathbb{N} \). Then, applying the inequality above with \( f = f_1 \cdots f_n \) and \( g = f_{n+1} \), we have
\[
\|f_1 \cdots f_{n+1}\|_r^r \leq \|f_1 \cdots f_n\|_{p}^r \|f_{n+1}\|_{p_{n+1}}^r.
\]
Now, by the induction hypothesis, taking \( p = \left( \sum_{i=1}^{n} \frac{1}{p_i} \right)^{-1} \), we have the inequality for the norm of \( \|f_1 \cdots f_n\|_{p}^r \), and the generalization follows as desired.

Q.E.D.
Exercise 8.8

Prove the following integral version of Minkowski’s inequality for $1 \leq p < \infty$:

$$\left[ \int \left| \int f(x,y) \, dx \right|^p \, dy \right]^{\frac{1}{p}} \leq \int \left[ \int |f(x,y)|^p \, dy \right]^{\frac{1}{p}} \, dx.$$  

Solution.

Suppose first that $p = 1$. Then,

$$\int \left| \int f(x,y) \, dx \right| \, dy \leq \int \int |f(x,y)| \, dx \, dy = \int \int |f(x,y)| \, dy \, dx,$$

where the last equality is achieved using Fubini’s theorem for nonnegative measurable functions.

Now, let $1 < p < \infty$. If the left-hand side is zero, we have nothing more to prove. Now, define $F(y) = \int |f(z,y)| \, dz$ and observe

$$\int \left| \int f(x,y) \, dx \right|^p \, dy \leq \int \int (F(y))^{p-1} |f(x,y)| \, dx \, dy$$

$$= \int \int (F(y))^{p-1} |f(x,y)| \, dy \, dx$$

$$\leq \int \left( \int |f(x,y)|^p \, dy \right)^{\frac{1}{p}} \cdot \left( \int (F(y))^p \, dy \right)^{\frac{p-1}{p}} \, dx.$$  

Notice that $\int (F(y))^p \, dy$ is, in fact, a constant (so any powers of it doesn’t change the fact that it is a constant), so we can pull it out of the integral by the integral’s linearity, divide by it since we observed above that it is positive, and we get

$$\left[ \int \left| \int f(x,y) \, dx \right|^p \, dy \right]^{\frac{1}{p}} \leq \int \left[ \int |f(x,y)|^p \, dy \right]^{\frac{1}{p}} \, dx,$$

as was to be shown.

Q.E.D.