Problem Set #2
Math 471 – Real Analysis
Assignment: Chapter 2, #9,11,13,18

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September 12, 2012
Exercise 2.9

Let $C$ be a curve with parametric equations $x = \varphi(t)$ and $y = \psi(t)$, $a \leq t \leq b$.

(a) If $\varphi$ and $\psi$ are of bounded variation and continuous, show that $L = \lim_{|\Gamma| \to 0} l(\Gamma)$.

(b) If $\varphi$ and $\psi$ are continuously differentiable, show that

$$ L = \int_a^b \left( [\varphi'(t)]^2 + [\psi'(t)]^2 \right)^{1/2} dt. $$

Solution.

(a) In terms of limits, we can state the question in the following way: we wish to show that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|\Gamma| < \delta$ implies $|L - \ell(\Gamma)| < \varepsilon$.

Since $\varphi$ and $\psi$ are of bounded variation, $L$ is finite by Theorem (2.13). Since $L$ is the supremum of $\ell(\Gamma)$ over all partitions, we can fix a partition $\Gamma_1$ in the interval $[a, b]$ such that $|L - \ell(\Gamma_1)| < \varepsilon$. Now, since $\varphi$ and $\psi$ are continuous on a compact set, we know they are uniformly continuous; that is, for any $\varepsilon > 0$, we can find $\delta_\varphi > 0$ and $\delta_\psi > 0$ such that $|x - y| < \delta$ implies $|\varphi(x) - \varphi(y)| < \frac{\varepsilon}{n \sqrt{2}}$ and $|\psi(x) - \psi(y)| < \frac{\varepsilon}{n \sqrt{2}}$, where $\delta = \min\{\delta_\varphi, \delta_\psi\}$. Then, for any $|\Gamma| < \delta$, let $\Gamma_2 = \Gamma_1 \cup \Gamma$ so that $|\Gamma_2| < \delta$ and $|L - \ell(\Gamma_2)| < \varepsilon$. Thus,

$$ |L - \ell(\Gamma)| \leq |L - \ell(\Gamma_1)| + |\ell(\Gamma_1) - \ell(\Gamma)| $$

$$ \leq \varepsilon + \varepsilon $$

$$ = 2\varepsilon. $$

Since $\Gamma$ and $\varepsilon > 0$ are arbitrary, the result follows.

(b) Using the first part of the problem, we have:

$$ L = \lim_{|\Gamma| \to 0} \ell(\Gamma) $$

$$ = \lim_{n \to \infty} \sum_{i=1}^{n} \left( [\varphi(x_i) - \varphi(x_{i-1})]^2 + [\psi(x_i) - \psi(x_{i-1})]^2 \right)^{1/2} $$

$$ = \lim_{n \to \infty} \sum_{i=1}^{n} \left( [\varphi'(t)]^2 + [\psi'(t)]^2 \right)^{1/2} (x_i - x_{i-1}) $$

$$ = \int_a^b \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2} dt $$

from the Mean Value Theorem.

Q.E.D.
EXERCISE 2.11

Show that \( \int_a^b f \, d\varphi \) exists if and only if given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( |R_\Gamma - R_{\Gamma'}| < \varepsilon \) if \( |\Gamma|, |\Gamma'| < \delta \).

SOLUTION.

Suppose \( \int_a^b f \, d\varphi \) exists and equals \( I \). Then for any given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( |I - R_\Gamma| < \frac{\varepsilon}{2} \) for \( |\Gamma| < \delta \). Take \( |\Gamma|, |\Gamma'| < \delta \), then we have

\[
|R_\Gamma - R_{\Gamma'}| \leq |R_\Gamma - I| + |I - R_{\Gamma'}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Hence, existence of the Riemann-Stieltjes integral of \( f \) with respect to \( \varphi \) implies given any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that for any partitions \( \Gamma, \Gamma' \), if \( |\Gamma|, |\Gamma'| < \delta \), then \( |R_\Gamma - R_{\Gamma'}| < \varepsilon \).

Conversely, suppose that for any given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for any partitions \( \Gamma, \Gamma' \), if \( |\Gamma|, |\Gamma'| < \delta \), then \( |R_\Gamma - R_{\Gamma'}| < \varepsilon \). To that end, take \( \Gamma_n = \{a + i(b-a)/n\}_{i=0}^n \). Define

\[
S_n = \sum_{i=1}^n f(\xi_i)(\varphi(x_i) - \varphi(x_{i-1})),
\]

a Riemann-Stieltjes sum with respect to \( \varphi \). So, given \( \varepsilon > 0 \), take \( \delta > 0 \) such that \( |\Gamma_1|, |\Gamma_2| < \delta \) implies \( |R_{\Gamma_1} - R_{\Gamma_2}| < \frac{\varepsilon}{2} \). Take \( m, n \) so large that \( \frac{(b-a)}{m} < \delta \) and \( \frac{(b-a)}{n} < \delta \), so that \( |\Gamma_m|, |\Gamma_n| < \delta \), thus we have \( |S_m - S_n| < \frac{\varepsilon}{2} \). Since \( \varepsilon > 0 \) is arbitrary, this means \( S_n \) is a Cauchy sequence, hence converges. Define \( I = \lim_{n \to \infty} S_n \) and let \( \Gamma \) be a partition of \([a, b]\) with \( |\Gamma| < \delta \). Then,

\[
|R_\Gamma - I| \leq |R_\Gamma - S_n| + |S_n - I| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Hence, \( f \) is Riemann-Stieltjes integrable with respect to \( \varphi \).

Q.E.D.
**Exercise 2.13**

*Prove theorem (2.16).*

**Solution.**

This problem asks us to check the linearity of the Riemann-Stieltjes integral. To that end, we'll check the linearity in two steps; first, we'll check linearity in the integrand, and then we'll check linearity of the function we're integrating against. So, let \( c_1, c_2 \in \mathbb{R} \), and let \( A = c_1 \int_a^b f_1 \, d\varphi \) and \( B = c_2 \int_a^b f_2 \, d\varphi \) exist. We wish to show \( \int_a^b (c_1 f_1 + c_2 f_2) \, d\varphi \) exists and relate it to the functions we already know. For simplicity, let \( I = A + B \) and \( h = c_1 f_1 + c_2 f_2 \), then we have:

\[
\left| I - \sum_{i=1}^{n} h(\xi_i)(\varphi(x_i) - \varphi(x_{i-1})) \right| = \left| I - \sum_{i=1}^{n} [(c_1 f_1)(\xi_i) + (c_2 f_2)(\xi_i)](\varphi(x_i) - \varphi(x_{i-1})) \right|
\]

\[
= \left| I - \sum_{i=1}^{n} (c_1 f_1)(\xi_i)(\varphi(x_i) - \varphi(\xi_{i-1})) \right| - \sum_{i=1}^{n} (c_2 f_2)(\xi_i)(\varphi(x_i) - \varphi(x_{i-1}))
\]

\[
\leq \left| A - \sum_{i=1}^{n} (c_1 f_1)(\xi_i)(\varphi(x_i) - \varphi(\xi_{i-1})) \right|
\]

\[
+ \left| B - \sum_{i=1}^{n} (c_2 f_2)(\xi_i)(\varphi(x_i) - \varphi(x_{i-1})) \right|
\]

\[
= \left| A - c_1 \sum_{i=1}^{n} f_1(\xi_i)(\varphi(x_i) - \varphi(\xi_{i-1})) \right|
\]

\[
+ \left| B - c_2 \sum_{i=1}^{n} f_2(\xi_i)(\varphi(x_i) - \varphi(x_{i-1})) \right|
\]

\[
< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Notice that in the third line, we demonstrate the equality \( \int_a^b cf \, d\varphi = c \int_a^b f \, d\varphi \) for \( c \in \mathbb{R} \). Hence, we've proven linearity in the integrand portion of the Riemann-Stieltjes integral.

For the other part of linearity, we follow similar steps as above, letting \( c_1, c_2 \in \mathbb{R} \) and
\[ A = c_1 \int_a^b f d\varphi_1 \text{ while } B = c_2 \int_a^b f d\varphi_2 \text{ and } I = A + B. \] Now, we have:

\[
\left| I - \sum_{i=1}^{n} f(\xi_i)((c_1\varphi_1 + c_2\varphi_2)(x_i) - (c_1\varphi_1 + c_2\varphi_2)(x_{i-1})) \right|
\]

\[
= \left| I - \sum_{i=1}^{n} f(\xi_i)((c_1\varphi_1)(x_i) + (c_2\varphi_2)(x_i) - (c_1\varphi_1)(x_{i-1}) - (c_2\varphi_2)(x_{i-1})) \right|
\]

\[
= \left| I - \sum_{i=1}^{n} f(\xi_i)((c_1)(\varphi_1(x_i) - \varphi(x_{i-1})) + c_2(\varphi_2(x_i) - \varphi_2(x_{i-1}))) \right|
\]

\[
\leq \left| A - c_1 \sum_{i=1}^{n} f(\xi_i)(\varphi_1(x_i) - \varphi(x_{i-1})) \right| + \left| B - c_2 \sum_{i=1}^{n} f(\xi_i)(\varphi_2(x_i) - \varphi_2(x_{i-1})) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\]

\[ = \varepsilon. \]

Thus, we’ve established the identity

\[
\int_a^b f d(c_1\varphi_1 + c_2\varphi_2) = c_1 \int_a^b f d\varphi_1 + c_2 \int_a^b f d\varphi_2
\]

provided the two integrals on the right exist.

Q.E.D.
Exercise 2.18

Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be a power series. Show that if \( \sum |a_k| < \infty \), then \( f(z) \) is of bounded variation on every radius of the circle \( |z| = 1 \). If, for example, the radius is \( 0 \leq x \leq 1 \) and the \( a_k \) are real, then \( f(x) = \sum a_k^+ x^k - \sum a_k^- x^k \).

Solution.

The hint tells us that, under certain conditions, we can write the function as a difference of two increasing functions. We can follow a similar path. First, consider the \( a_k \)'s as being complex numbers. Then we can rewrite the coefficients as \( a_k = b_k + ic_k \) where \( b_k, c_k \in \mathbb{R} \).

Now we have:

\[
\begin{align*}
  f(z) &= \sum_{k=1}^{\infty} a_k (re^{i\theta})^k \\
        &= \sum_{k=1}^{\infty} (b_k + ic_k)(re^{i\theta})^k \\
        &= \sum_{k=1}^{\infty} b_k (re^{i\theta})^k + i \sum_{k=1}^{\infty} c_k (re^{i\theta})^k \\
        &\text{of bounded variation by the hint} & \text{of bounded variation}
\end{align*}
\]

Now, we invoke the theorem that states a complex-valued function is of bounded variation if and only if its real and imaginary parts are of bounded variation. Hence, \( f(z) \) is of bounded variation and we are done.

Q.E.D.