Problem Set #4
Math 471 – Real Analysis
Assignment: Chapter 3, #12, 15, 20

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Exercise 3.12

If $E_1$ and $E_2$ are measurable subsets of $\mathbb{R}^1$, show that $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$. The convention is to interpret $0 \cdot \infty = 0$.

Solution.

We will handle this problem in three subsequent steps. Observe that if $|E_1|, |E_2| = \infty$, or if $0 < |E_1| < \infty$ and $|E_2| = \infty$, then the result holds trivially. Thus, assume that $|E_1|, |E_2| < \infty$.

First, let us show that if $E_1, E_2$ are open sets in $\mathbb{R}$, then $E_1 \times E_2$ has the desired properties. To that end, since $E_1$ and $E_2$ are open, we can write them as a union of disjoint open intervals, say $E_1 = \bigcup_{k=1}^{\infty} I_k$ and $E_2 = \bigcup_{j=1}^{\infty} J_j$. Then we have

$$E_1 \times E_2 = \bigcup_{k=1}^{\infty} I_k \times \bigcup_{j=1}^{\infty} J_j = \bigcup_{k,j=1}^{\infty} (I_k \times J_j).$$

Thus, $E_1 \times E_2$ is an open set in $\mathbb{R}^2$, hence is measurable. Moreover, since the intervals are disjoint, we have

$$|E_1 \times E_2| = \left| \bigcup_{k=1}^{\infty} I_k \times \bigcup_{j=1}^{\infty} J_j \right| = \sum_{k,j=1}^{\infty} |I_k||J_j| = |E_1||E_2|. $$

Therefore, for open, measurable subsets of $\mathbb{R}$, we have the desired property.

Now let us consider the more general $G_\delta$-sets. This is a decreasing sequence of open sets by intersection, thus we can write $E_1 \times E_2$ as a decreasing product of open sets, hence $E_1 \times E_2$ is a $G_\delta$ and is therefore measurable. Furthermore, we have

$$|E_1 \times E_2| = \left| \lim_{k \to \infty} G_k \times H_k \right| = \lim_{k \to \infty} |G_k \times H_k| = \lim_{k \to \infty} |G_k||H_k| = |E_1||E_2|. $$

Now we will show that the product of a set with measure zero and any other set will have measure zero in $\mathbb{R}^2$. Without loss of generality, assume $|E_1| = 0$ and consider the case where $|E_2| < \infty$. Then we can find open covers $C_1 = \{I_k\}_{k=1}^{\infty}$ and $C_2 = \{J_j\}_{j=1}^{\infty}$ with $E_1 \subseteq \bigcup_{k=1}^{\infty} I_k$ and $E_2 \subseteq \bigcup_{j=1}^{\infty} J_j$. Note also that, given $\varepsilon, \varepsilon_0 > 0$, these open covers satisfy the additional condition

$$\left| \bigcup_{k=1}^{\infty} I_k \right| = \sum_{k=1}^{\infty} |I_k| < \frac{\varepsilon}{|E_2| + \varepsilon_0} \quad \text{and} \quad \left| \bigcup_{j=1}^{\infty} J_j \right| = \sum_{j=1}^{\infty} |J_j| < |E_2| + \varepsilon_0.$$

Now we have $C_1 \times C_2 = \{I_k \times J_j : k, j \in \mathbb{Z}^+\}$ and this is a countable cover of $E_1 \times E_2$. Considering the measure of $C_1 \times C_2$, we have

$$\sum_{k=1,j=1}^{\infty} |I_k \times J_j| = \sum_{k=1,j=1}^{\infty} |I_k||J_j| < \frac{\varepsilon}{|E_2| + \varepsilon_0}(|E_2| + \varepsilon_0) = \varepsilon.$$

Thus, $E_1 \times E_2$ is measurable and has measure zero. In the case that $|E_2| = \infty$, partition $E_2$ into sets such as $E_k = (((-k-1), -k] \cup [k, k+1)) \cap E$ for $k \geq 0$. Then clearly $\bigcup_{k=0}^{\infty} E_k = E$. 

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and $|E_k| \leq 2 < \infty$ for all $k$. Then by the preceding argument, each $E_1 \times E_k$ will have measure zero, and the countable union of sets of measure zero is of measure zero itself.

This finishes the proof since $E_1$ and $E_2$ are measurable if and only if they can be represented by a $G_\delta$-set minus a set of measure zero. Thus, observe that if $E_1 = H_1 - Z_1$ and $E_2 = H_2 - Z_2$, then

$$E_1 \times E_2 = (H_1 - Z_1) \times (H_2 - Z_2) = (H_1 \times H_2) - ((Z_1 \times H_2) \cup (H_1 \times Z_2)).$$

Hence, $E_1 \times E_2$ is measurable and has measure $|E_1||E_2|$.

Q.E.D.
Exercise 3.15

If $E$ is measurable and $A$ is any subset of $E$, show that $|E| = |A| + |E - A|_e$.

Solution.

For simplicity, we will use the complement with respect to $E$. Let $G$ be any open set containing $A$, where $A \subseteq E$. Then $G^C$ is closed and $G^C \subseteq A^C$, thus we have

$$|G| + |E - A|_i = |G| + |A^C|_i \geq |G| + |G^C| = |G \cup G^C| = |E|.$$ 

Taking the infimum over all open covers of $A$, we have

$$|A| + |A^C|_i \geq |E|.$$ 

Conversely, if $F$ is closed and $F \subseteq A^C$, then $F^C \supseteq A$, so

$$|A|_i + |F| \leq |F^C| + |F| = |F^C \cup F| = |E|.$$ 

Taking the supremum over all closed sets contained in $A^C$, we have

$$|A|_i + |A^C|_i \leq |E|.$$ 

Therefore, $|E| \leq |A|_i + |E - A|_i \leq |E|$, hence we have $|E| = |A|_i + |E - A|_i$.

Realizing that this is the exact opposite of what was asked, we show the solution to the problem below (in a similar fashion).

Let $F$ be any closed set contained in $A$, where $A \subseteq E$. Then $F^C$ is open and $F^C \supseteq A^C$, thus we have

$$|F| + |A^C|_e \leq |F| + |F^C| = |E|.$$ 

Taking the supremum of all closed subsets of $A$, we have

$$|A| + |A^C|_e \leq |E|.$$ 

Conversely, if $G$ is open and $G \supseteq A^C$, then $G^C \subseteq A$, so

$$|A|_i + |G| \geq |G^C| + |G| = |E|.$$ 

Taking the infimum over all open sets containing $A^C$, we have

$$|A|_i + |A^C|_e \geq |E|.$$ 

Therefore, $|E| \leq |A|_i + |A^C|_e \leq |E|$ which implies $|E| = |A|_i + |A^C|_e = |A|_i + |E - A|_e$.

Finally, this leaves us with the identity

$$|A|_e + |E - A|_i = |E| = |A|_i + |E - A|_e.$$

Q.E.D.
Exercise 3.20

Show that there exists disjoint $E_1, E_2, \ldots$, such that

$$\left| \bigcup_{k=1}^{\infty} E_k \right|_e < \sum_{k=1}^{\infty} |E_k|_e.$$ 

Solution.

Consider a nonmeasurable subset $E$ of $[0,1]$ with disjoint rational translates; observe that $|E|_e > 0$ since otherwise it would be measurable by example 2 on p. 37. Enumerate the rational numbers in $[0,1]$ by $\{q_k\}_{k=1}^\infty$. Then

$$\bigcup_{k=1}^{\infty} E + q_k \subseteq [0,2].$$

By monotonicity of outer measure, we have

$$\left| \bigcup_{k=1}^{\infty} E + q_k \right|_e \leq ||[0,2]|_e = 2.$$ 

Now, considering the sum,

$$\left| \bigcup_{k=1}^{\infty} E + q_k \right|_e \leq \sum_{k=1}^{\infty} |E + q_k|_e = \sum_{k=1}^{\infty} |E|_e,$$

since $|E|_e > 0$, there exists a positive integer $N$ such that $N|E|_e > 2$ by the Archimedean property, hence,

$$\left| \bigcup_{k=1}^{\infty} E + q_k \right|_e \leq 2 < \sum_{k=1}^{N} |E + q_k|_e = \sum_{k=1}^{N} |E|_e < \sum_{k=1}^{\infty} |E|_e.$$ 

Note that we relied on translation invariance of outer measure in this proof so that $|E + q_k|_e = |E|_e$.

Q.E.D.