Problem Set #7
Math 471 – Real Analysis
Assignment: Chapter 4 #15, 16, 17

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Exercise 4.15

Let \( \{f_k\}_{k=1}^{\infty} \) be a sequence of measurable functions defined on a measurable set \( E \) with \( |E| < \infty \). If \( |f_k(x)| \leq M_x < \infty \) for all \( k \) for each \( x \in E \), show that given \( \varepsilon > 0 \), there is a closed set \( F \subseteq E \) and a finite \( M \) such that \( |E - F| < \varepsilon \) and \( |f_k(x)| \leq M \) for all \( k \) and all \( x \in E \).

Solution.

Let \( \varepsilon > 0 \) and \( f(x) = \sup_{k \in \mathbb{N}} f_k(x) \). We know that since each \( f_k \) is measurable, \( f \) is measurable, and we know that \( f(x) \leq M_x \) for all \( x \in E \). Thus, we can use Lusin’s Theorem to get a closed set \( F \subseteq E \) such that \( f|_F \) is continuous and \( |E - F| < \varepsilon \). Since \( |E| < \infty \), we can find a compact set \( F^* \subseteq F \) such that \( |E - F^*| < \varepsilon \) (to see this, we can simply take a closed ball of radius \( R \) such that \( |E - B_R(0)| < \varepsilon \) and take the intersection with \( F \)). Then, since \( f \) is continuous relative to \( F \), it is continuous relative to \( F^* \), hence attains its maximum, i.e., there is a constant \( M \) such that \( f(x) \leq M \) for all \( x \in E \).

Q.E.D.
Exercise 4.16

Prove that \( f_k \to f \) in measure on \( E \) if and only if given \( \varepsilon > 0 \), there exists a \( K \) such that \(|\{|f - f_k| > \varepsilon\}| < \varepsilon \) if \( k > K \). Give an analogous Cauchy criterion.

Solution.

Let \( f_k \xrightarrow{m} f \). Then let \( \delta, \varepsilon > 0 \). By definition, we have \(|\{|x \in E : |f(x) - f_k(x)| > \delta\}| < \varepsilon \). Letting \( \delta = \varepsilon \) gives us the desired result, thus proving one direction.

Conversely, suppose given \( \delta, \varepsilon > 0 \), there exists a \( K_\delta \) and \( K_\varepsilon \) such that \(|\{|f - f_k| > \delta\}| < \delta \) if \( k > K_\delta \) and \(||f - f_k| > \varepsilon| < \varepsilon \) if \( k > K_\varepsilon \). Let \( \gamma = \min\{\delta, \varepsilon\} \) and \( k > \max\{K_\delta, K_\varepsilon\} \). Then,

\[
\{x \in E : |f(x) - f_k(x)| > \varepsilon\} \subseteq \{x \in E : |f(x) - f_k(x)| > \gamma\},
\]

which implies

\[
|\{|f - f_k| > \varepsilon\}| \leq |\{|f - f_k| > \gamma\}| < \gamma \leq \delta,
\]

thus, \( f_k \xrightarrow{m} f \).

An analogous Cauchy criterion is as follows; \( f_k \xrightarrow{m} f \) if and only if for every \( \varepsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( m, n > N \) implies \(|\{|f_n - f_m| > \varepsilon\}| < \varepsilon \).

Q.E.D.
Exercise 4.17

Suppose that \( f_k \xrightarrow{m} f \) and \( g_k \xrightarrow{m} g \) on \( E \). Show that \( f_k + g_k \xrightarrow{m} f + g \) on \( E \) and, if \( |E| < \infty \), that \( f_k g_k \xrightarrow{m} fg \) on \( E \). If, in addition, \( g_k \to g \) on \( E \), \( g \neq 0 \) a.e., and \( |E| < \infty \), show that \( \frac{f_k}{g_k} \xrightarrow{m} \frac{f}{g} \) on \( E \).

Solution.

Let \( f_k \xrightarrow{m} f \) and \( g_k \xrightarrow{m} g \). Then,

\[
|\{x \in E : |(f + g)(x) - (f_k + g_k)(x)|\}| = |\{x \in E : |f(x) - f_k(x) + g(x) - g_k(x)| > \varepsilon \}| \\
\leq |\{x \in E : |f(x) - f_k(x)| + |g(x) - g_k(x)| > \varepsilon \}| \\
\leq |\{x \in E : |f(x) - f_k(x)| > \frac{\varepsilon}{2}\}| \\
+ |\{x \in E : |g(x) - g_k(x)| > \frac{\varepsilon}{2}\}| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon.
\]

Note that we get the second inequality from observing

\[
\{x \in E : |f(x) - f_k(x)| + |g(x) - g_k(x)| > \varepsilon \} \\
\subseteq \left\{ x \in E : |f_k(x) - f(x)| > \frac{\varepsilon}{2} \right\} \cup \left\{ x \in E : |f_k(x) - g(x)| > \frac{\varepsilon}{2} \right\}.
\]

Now, let \( |E| < \infty \). Similar to the above, note that

\[
\{x \in E : |f(x) - f_k(x)||g_k(x) - g(x)| < \varepsilon \} \\
\subseteq \left\{ x \in E : |f(x) - f_k(x)| < \sqrt{\frac{\varepsilon}{3}} \right\} \cup \left\{ x \in E : |g(x) - g_k(x)| < \sqrt{\frac{\varepsilon}{3}} \right\}.
\]

Then, from the hint, note that \( f \) and \( g \) are bounded on sets whose complements relative to \( E \) have small measure, thus, we can fix \( M \) such that \( |\{x \in E : |f(x)| > M\}| < \frac{\varepsilon}{6} \) and similarly for \( g \). Now we can choose \( K \) large enough so that, for \( k > K \), we have

\[
\left| \left\{ f_k - f \right\} \right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left| \left\{ g_k - g \right\} \right| < \frac{\varepsilon}{3}
\]

and

\[
\left| \left\{ f_k - f \right\} \right| < \frac{\varepsilon}{3M} \quad \text{and} \quad \left| \left\{ g_k - g \right\} \right| < \frac{\varepsilon}{3M}.
\]
Then we have

\[
|\{ |f_k g_k - f g| > \varepsilon \}| \leq \left| \left\{ |f - f_k||g - g_k| > \frac{\varepsilon}{3} \right\} \right| + \left| \left\{ ||f||g - g_k| > \frac{\varepsilon}{3} \right\} \right| + \left| \left\{ |g||f - f_k| > \frac{\varepsilon}{3} \right\} \right|
\]

\[
\leq \left| \left\{ |f - f_k| > \frac{\varepsilon}{3} \right\} \cup \left\{ |g - g_k| > \frac{\varepsilon}{3} \right\} \right| + \left| \left\{ |f| > M \right\} \cup \left\{ |g - g_k| > \frac{\varepsilon}{3M} \right\} \right|
\]

\[
+ \left| \left\{ |g| > N \right\} \cup \left\{ |f - f_k| > \frac{\varepsilon}{3N} \right\} \right|
\]

\[
< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon
\]

For the last part, simply observe that \( \frac{f_k}{g_k} = f_k \frac{1}{g_k} \), hence, if we set \( h_k = \frac{1}{g_k} \), we see that \( h_k \to \frac{1}{g} \) a.e. as \( k \to \infty \). By Proposition 4.21, this implies \( h_k \overset{m}{\to} \frac{1}{g} \), then we see \( \frac{f_k}{g_k} \overset{m}{\to} \frac{f}{g} \) follows as desired from the previous result regarding products.

Q.E.D.