Exercise 1 (Each part 1 mark). Consider a class of 50 students. For each student, a fair six-sided die will be rolled to determine the student’s final grade. If the die shows 6, the grade is 90. If the die shows any other number, the grade is 40. Let $X_i$ be the grade of the $i$-th student. Let $Z = \frac{1}{50} \sum_{i=1}^{50} X_i$ be the class average. 
(a) What is the expected grade of the $i$-th student?
(b) If only 8 students roll a 6, what is the class average?
(c) What is the expected class average?

Exercise 2 (Each part 1 mark). Consider a coin where the probability of heads is $p$. Flip the coin $n$ times. Define $X_i = 1$ if $i$-th flip is heads, 0 otherwise. Define $Y = \sum_{i=1}^{n} X_i$.
(a) Express the event that there are exactly $k$ heads in terms of $Y$. Hint: If there are exactly $k$ heads, what is the value of $Y$?
(b) Find $E(Y)$.

Exercise 3 (Each part 1 mark). The variance of a random variable $X$ is defined to be
$$\text{Var}(X) = E((X - E(X))^2).$$
It is the average squared-distance between $X$ and its average $E(X)$.
(a) Use the properties of expectation to prove $\text{Var}(X) = E(X^2) - (E(X))^2$.
(b) Let $X$ be the number shown after rolling a fair six-sided die. Find $E(X^2)$ and $\text{Var}(X)$.

Exercise 4 (Each part 0.25 mark.). Which of the following necessarily imply a violation of the no-arbitrage assumption? Assume $T > 0$ and $\epsilon > 0$.
(a) A portfolio which has zero value today, always non-negative value at $T$, and positive value at $T$ for some sample outcomes $\omega$ with $P(\{\omega\}) > 0$.
(b) A portfolio which has zero value today and expected positive value at $T$.
(c) A portfolio which has value $-\epsilon$ today and zero value at $T$.
(d) A portfolio which has value $\epsilon$ today and expected positive value at $T$.
(e) A portfolio which has zero value today and value $\epsilon$ at $T$.
(f) A portfolio which has zero value today and positive value at $T$ for some sample outcomes with positive probability.
(g) A portfolio which has zero value today, always non-negative value at $T$, and positive value at $T$ for some sample outcomes.
(h) A portfolio which has zero value today, always non-negative value at $T$, and expected positive value at $T$. 
Exercise 5 (Each part 2 marks). We showed in class (and in the lecture notes) that the no-arbitrage principle implies the monotonicity principle. Consider the

**Strong Monotonicity Principle.** Let $A$ and $B$ be portfolios and let $T > t$, where $t$ is the current time. If $V^A(T) \geq V^B(T)$ with probability one, and $V^A(T) > V^B(T)$ with positive probability, then $V^A(t) > V^B(t)$.

(a) Show that the no-arbitrage principle implies the strong monotonicity principle.
(b) Show that the strong monotonicity principle implies the monotonicity principle.
(c) Show that the strong monotonicity principle implies the no-arbitrage principle. Hint: If $A$ is an arbitrage portfolio, apply the monotonicity principle to $A$ and an empty portfolio $B$ to deduce a contradiction.
Solutions.

1. 
(a) \( R(X_i) = \{40, 90\} \).

\[
\mathbb{E}(X_i) = \sum_{k \in R(X_i)} kP(X_i = k) = 40P(X_i = 40) + 90P(X_i = 90) = 40 \cdot \frac{5}{6} + 90 \cdot \frac{1}{6}
\]

(b) \( Z = \frac{1}{50} (8(90) + 42(40)) \)

(c) 
\[
\mathbb{E}(Z) = \mathbb{E} \left( \frac{1}{50} \sum_{i=1}^{50} X_i \right) = \frac{1}{50} \mathbb{E} \left( \sum_{i=1}^{50} X_i \right) = \frac{1}{50} \sum_{i=1}^{50} \mathbb{E}(X_i)
\]

\[
= \frac{1}{50} \sum_{i=1}^{50} \left( 40 \cdot \frac{5}{6} + 90 \cdot \frac{1}{6} \right) = \frac{1}{50} \cdot 50 \cdot \left( 40 \cdot \frac{5}{6} + 90 \cdot \frac{1}{6} \right) = 40 \cdot \frac{5}{6} + 90 \cdot \frac{1}{6}
\]

2. 
(a) \( \{Y = k\} \). The marker will also accept \( Y = k \), without the brackets, even though this doesn’t technically represent an event.

(b) 
\[
\mathbb{E}(Y) = \mathbb{E} \left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \mathbb{E}(X_i).
\]

So we need to find \( \mathbb{E}(X_i) \) for arbitrary \( i \). Note \( R(X_i) = \{1, 0\} \), \( P(X_i = 1) = p \) and \( P(X_i = 1) = 1 - p \). Therefore

\[
\mathbb{E}(X_i) = \sum_{k \in R(X_i)} kP(X_i = k) = 1P(X_i = 1) + 0P(X_i = 0) = p.
\]

Therefore

\[
\mathbb{E}(Y) = \sum_{i=1}^{n} p = np.
\]

3. 
(a) The key is to realize \( \mathbb{E}(X) \) is a constant.

\[
\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2 - 2\mathbb{E}(X)X + (\mathbb{E}(X))^2)
\]

\[
= \mathbb{E}(X^2) - \mathbb{E}(2\mathbb{E}(X)X) + \mathbb{E}((\mathbb{E}(X))^2) = \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + (\mathbb{E}(X))^2
\]

\[
= \mathbb{E}(X^2) - (\mathbb{E}(X))^2
\]

(b) \( R(X^2) = \{1, 4, 9, 16, 25, 36\} \).

\[
\mathbb{E}(X^2) = \sum_{k \in R(X^2)} kP(X^2 = k)
\]

\[
= 1P(X^2 = 1) + 4P(X^2 = 4) + 9P(X^2 = 9) + 16P(X^2 = 16) + 25P(X^2 = 25) + 36P(X^2 = 36)
\]

\[
= 1(1/6) + 4(1/6) + 9(1/6) + 16(1/6) + 25(1/6) + 36(1/6) = \frac{91}{6}
\]
Remember from class (or recalculate) that $E(X) = 21/6 = 3.5$. By the result of (a),

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{105}{36} = 2.91666\ldots$$

4.

Note: Detailed explanations are given on the last page of the solutions.

(a) Implies a violation of no-arbitrage.
(b) Does not imply violation of no-arbitrage.
(c) Implies violation of no-arbitrage.
(d) Does not imply violation of no-arbitrage.
(e) Implies violation of no-arbitrage.
(f) Does not imply violation of no-arbitrage.
(g) Does not imply violation of no-arbitrage.
(h) Implies violation of no-arbitrage.

5.

(a)

Assume the no-arbitrage principle.

Assume (1) $V^A(T) \geq V^B(T)$ with probability one and (2) $V^A(T) > V^B(T)$ with positive probability. We want to show $V^A(t) > V^B(t)$. So we assume (3) $V^A(t) \leq V^B(t)$, and we show this leads to a contradiction.

Consider the portfolio $C$ consisting of $A$ minus $B$. For example, if $A$ consists of 5 shares of AAPL and $B$ consists of 3 shares of GOOGL, then $C$ consists of 5 shares of AAPL and −3 shares of GOOGL. Then (by our assumptions (3),(1),(2) [in that order]) we have

- $V^C(t) = V^A(t) - V^B(t) \leq 0$
- $V^C(T) = V^A(T) - V^B(T) \geq 0$ with probability one
- $V^C(T) = V^A(T) - V^B(T) > 0$ with positive probability

Therefore $C$ is an arbitrage portfolio. This contradicts the no-arbitrage principle. Done.

(b)

Assume the strong monotonicity principle.

Let $A$ and $B$ be portfolios and $T > t$, with $t$ the current time. Assume $V^A(T) \geq V^B(T)$ with probability one. We need to show that $V^A(t) \geq V^B(t)$. So we assume $V^A(t) < V^B(t)$ and show this leads to a contradiction.

Let $\epsilon = V^B(t) - V^A(t) > 0$. Let $A(\epsilon)$ be the portfolio equal to $A$ plus $\epsilon$ cash. Then $V^{A(\epsilon)}(T) = V^A(T) + \epsilon > V^B(T)$ with probability one. The strong monotonicity principle (applied to $A(\epsilon)$ and $B$) implies

$$V^{A(\epsilon)}(t) = V^A(t) + \epsilon > V^B(t),$$
But, since $\epsilon = V^B(t) - V^A(t) > 0$, the last inequality says

$$V^B(t) > V^B(t).$$

Contradiction. Done.

(c)

Assume the strong monotonicity principle.

We want to prove there are no arbitrage portfolios. So we assume there is an arbitrage portfolio, and we deduce a contradiction.

Call the arbitrage portfolio $A$. We know

- $V^A(t) \leq 0$
- $V^A(T) \geq 0$ with probability one.
- $V^A(T) > 0$ with positive probability.

Let $B$ be an empty portfolio. Then $V^B(t) = 0$ and $V^B(t) = 0$ with probability one. Therefore

(i) $V^A(t) \leq V^B(t)$

(ii) $V^A(T) \geq V^B(T)$ with probability one.

(iii) $V^A(T) > V^B(T)$ with positive probability.

By the strong monotonicity principle, (ii) and (iii) imply $V^A(t) > V^B(t)$. This contradicts (i). Done.

(a) Implies violation of no-arbitrage.
(b) Does not imply violation of no-arbitrage.
(c) Implies violation of no-arbitrage.
(d) Does not imply violation of no-arbitrage.
(e) Implies violation of no-arbitrage.
(f) Does not imply violation of no-arbitrage.
(g) Does not imply violation of no-arbitrage.
(h) Implies violation of no-arbitrage.

The explanations are not in order.

(a), (e). This is clearly an arbitrage portfolio.

(c). Consider portfolio $A$ which equals the given portfolio plus $\epsilon$ of cash. Then $V^A(0) = 0$ and $V^A(T) = \epsilon$ with probability one, assuming the cash is not invested. If the cash is invested, $V^A(T) > \epsilon$.

(b), (d), (h). $\mathbb{E}(V(T)) > 0$ means $\sum_{v \in R(V(T))} vP(V(T) = v) > 0$. This implies that there is some value $v$ such that $vP(V(T) = v) > 0$, hence $v > 0$ and $P(V(T) = v) > 0$. But it doesn’t rule out that there is some other value $v$ where $vP(V(T) = v) < 0$, hence $v < 0$ and $P(V(T) = v) > 0$.

(f). The portfolio could have negative value at other sample outcomes with positive probability.

(g). This one is subtle. It does not say the portfolio has “positive value at $T$ for some sample outcomes with positive probability.” The sample outcomes where the portfolio has positive value could have probability zero. This is a counter-intuitive situation that can come up when dealing with an infinite sample space. It doesn’t make much sense in practice. But in order for our mathematical model to be consistent, we cannot avoid such situations.